MATH105: Solutions to Practice Problems

6. When $n = 2$

$$2^n + n^2 = 4 + 4 = 8 < 9 = 3^2$$

So $2^n + n^2 < 3^n$ is true for $n = 2$. Now assume it is true for some $n \in \mathbb{N}$ with $n \geq 2$. Then using $(n+1)^2 \leq \frac{9}{4}n^2$,

$$2^{n+1} + (n+1)^2 \leq 2^n + 2^n + \frac{9}{4}n^2 = 2 \times (2^n + n^2) + \frac{1}{4}n^2 < 3(2^n + n^2) < 3 \cdot 3^n = 3^{n+1}.$$

So $2^n + n^2 < 3^n \Rightarrow 2^{n+1} + (n+1)^2 < 3^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 2$. So by induction, $2^n + n^2 < 3^n$ for all $n \in \mathbb{N}$ with $n \geq 2$.

7. When $n = 0$ we have $a_0 = 2 = 3^0 + 1$. So $a_n = 3^n + 1$ is true for $n = 0$. Now assume that $a_n = 3^n + 1$ for some $n \in \mathbb{N}$. Then $a_{n+1} = 3a_n - 2 = 3(3^n + 1) - 2 = 3^{n+1} + 1$. So by induction the formula $a_n = 3^n + 1$ holds for all $n \in \mathbb{N}$.

8. We have $\frac{1}{2} < a_0 = 1$. So $\frac{1}{2} \leq a_n \leq 1$ when $n = 0$. Now assume this holds for some $n \in \mathbb{N}$. We want to deduce it for $n + 1$. If $a_n \geq \frac{1}{2}$ then $3a_n + 1 \geq \frac{5}{2} > 0$. and $3a_n + 1 > a_n + 1$. So it is certainly true that

$$\frac{a_n + 1}{3a_n + 1} < 1$$

Since $3a_n + 1 > 0$ we have

$$\frac{1}{2} \leq \frac{a_n + 1}{3a_n + 1} \Leftrightarrow 3a_n + 1 \leq 2a_n + 2 \Leftrightarrow a_n \leq 1.$$

So

$$a_n \leq 1 \Rightarrow \frac{1}{2} < a_{n+1}$$

and

$$\frac{1}{2} \leq a_n \leq 1 \Rightarrow \frac{1}{2} \leq a_{n+1} \leq 1$$

So by induction $\frac{1}{2} \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

9. From the definition of multiplication, and $m \cdot 1 = m$ we have

$$m \cdot (n + 1) = m \cdot n + m = m \cdot n + m \cdot 1$$

So $m \cdot (n + p) = m \cdot n + m \cdot p$ is true for $p = 1$.

Now assume inductively that it is true for p. Then

$$m \cdot (n + (p + 1)) = m \cdot ((n + p) + 1) = m \cdot (n + p) + m \cdot 1 = (m \cdot n + m \cdot p) + m.$$

The first equality uses associativity of addition and the second uses the inductive definition of multiplication and the third uses the inductive hypothesis and $m \cdot 1 = m$. But then

$$(m \cdot n + m \cdot p) + m = m \cdot n + (m \cdot p + m) = m \cdot n + m \cdot (p + 1)$$

where the first equality uses associativity of multiplication and the second uses the inductive definition of multiplication. This completes the proof that

$$(m \cdot (n + p) = m \cdot n + m \cdot p) \quad \Rightarrow \quad (m \cdot (n + (p + 1)) = m \cdot n + m \cdot (p + 1))$$

So by induction

$$m \cdot (n + p) = m \cdot n + m \cdot p$$

for all m, n, $p \in \mathbb{Z}_+$.