# MATH105 Solutions to Practice Problems 10

**5.**

a) Because of the way the continued fraction expansion repeats, we need a number $x$ satisfying

$$x = \frac{1}{4+x}$$

that is

$$x^2 + 4x - 1 = 0.$$

This implies that

$$x = -2 \pm \sqrt{5}$$

Since all continued fractions with positive integers represent positive numbers, we must have $x = -2 + \sqrt{5}$.

b) This time we must have

$$x = \frac{1}{4 + \frac{1}{1+x}} = \frac{x+1}{4x+5}.$$

So

$$4x^2 + 4x - 1 = 0$$

and

$$x = \frac{-2 \pm \sqrt{8}}{4} = \frac{-1 \pm \sqrt{2}}{2}$$

and again we need to take the positive root. So $x = (-1 + \sqrt{2})/2$

**10.**

a) One could use calculus, but it is not necessary because if $x < y$ then $x^3 < y^3$ and hence $x^3 + 2x + 5 < y^3 + 2y + 5$. If using calculus, then $f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$, and hence $f$ is strictly increasing.

b) There are no integer solutions to $f(x) = 0$ because $f(-2) = -7$ and $f(-1) = 2$. So $f(n) < 0$ for all $n \in Z$ with $n \leq -2$ and $f(n) > 0$ for all $n \in \mathbb{Z}$ with $n \geq -1$. Suppose

$$\frac{p^3}{q^3} + 2\frac{p}{q} + 5 = 0$$

for $p \in Z$ and $q \in \mathbb{Z}_+$. We can assume the g.c.d of $p$ and $q$ is one and then $q \geq 2$ because there are no integer solutions to $f(x) = 0$Then multiplying by $q^3$ we have

$$p^3 + 2pq^2 + 5q^3 = 0$$

This can be rewritten as

$$p^3 = -q^2(2p + 5q)$$

Let $k$ be any prime factor of $q$. There is at least one because $q \geq 2$.Then $k|p^3$. Hence by unique factorisation, $k|p$ and $k$ is a factor of both $p$ and $q$, giving a contradiction.

c) The set $A = \{x \in \mathbb{Q} : x^3 + 2x + 5 < 0\}$ is a Dedekind cut because it has no maximal element, $0 \notin A$ and $x \in A \wedge y < x \Rightarrow f(y) < f(x) < 0 \Rightarrow y \in A$.

**15.**

a) For $f(x) = x^3 - 12x + 1$,

$$f(-4) = -15 < 0, \quad f(-3) = 10 > 0, \quad f(0) = 1 > 0, \quad f(1) = -10 < 0,$$

$$f(3) = -8 < 0, \quad f(4) = 15 > 0.$$

Applying the intermediate value theorem to $f$ on each of the intervals $[-4,-3]$, $[0,1]$ and $[3,4]$, we see that $f$ has a zero in each of the intervals $(-4,-3)$, $(0,1)$ and $(3,4)$. Also $f'(x) = 3x^2 - 12 = 3(x^2-4) = 0 \leftrightarrow x = \pm 2$. Also $f'(x) > 0$ if $x \in (-\infty,-2) \cup [(2,\infty)$ and $f'(x) < 0$ on $(-2,2)$. So $f$ is strictly increasing on each of the intervals $(-\infty,-2]$ and on $[2,\infty)$, and strictly decreasing on $[-2,2]$. In particular $f$ is strictly increasing on each of the intervals $[-4,-3]$ and $[3,4]$ and strictly decreasing on $[0,1]$. So because of the values of $f$ that have been computed, $f$ must have a zero in each of the intervals $(-4,-3)$, $(3,4)$ and $(0,1)$.

b) The Dedekind cuts can be expressed as

$$A_1 = \{x \in \mathbb{Q} : f(x) < 0 \wedge x < -3\}, \quad A_2 = \{x \in \mathbb{Q} : x < -3\} \cup \{x \in \mathbb{Q} : f(x) > 0 \wedge x < 1\},$$

$$A_3 = \{x \in \mathbb{Q} : x < 3 \vee f(x) < 0\}$$

In each case, $x \in A_j \wedge y < x \Rightarrow y \in A_j$ and $5 \notin A_j$ and $A_j$ has no maximal element. Full proof of $A_j$ not having a maximal element is not required.