MATH105 Problem Sheet 7: Solutions to Problems 5-9

5.

a) The inverse does not exist, because f is not surjective. In fact, since (as shown in lectures)

$$1 + x + x^2 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\mathrm{Im}(f) = [3/4, \infty)$.

b) This time

$$y = 1 + x + x^2 \quad \Leftrightarrow \quad y - \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 \quad \Leftrightarrow \quad x = -\frac{1}{2} + \sqrt{y - \frac{3}{4}}$$

So $f^{-1} : [3/4, \infty) \to [-1/2, \infty)$ is defined by

$$f^{-1}(y) = -\frac{1}{2} + \sqrt{y - \frac{3}{4}}.$$

c) We have

$$y = \frac{1}{x+1} \quad \Leftrightarrow \quad xy + y = 1 \quad \Leftrightarrow \quad x = \frac{1}{y} - 1$$

from which we see that $x > -1 \Leftrightarrow y > 0$. So $f^{-1} : (0, \infty) \to (-1, \infty)$ does exist and is defined by

$$f^{-1}(y) = \frac{1}{y} - 1.$$

6. Let P, C and T denote the sets of plots growing, potatoes, courgettes and tomatoes respectively.

a) The inclusion-exclusion principle for three sets says that

$$|P \cup C \cup T| = |P| + |C| + |T| - |P \cap C| - |P \cap T| - |C \cap T| + |P \cap C \cap T|,$$

that is

$$20 = 15 + 12 + 10 - 8 - 7 - 6 + |P \cap C \cap T|$$

that is $|P \cap C \cap T| = 4$, that is, 4 plots grow all 3 crops.

b) The set of plots growing only potatoes is $P \setminus (C \cup T)$ and

$$|P \setminus (C \cup T)| = |P| - |P \cap (C \cup T)|.$$

Applying the inclusion-exclusion principle for two sets to the sets $P \cap C$ and $P \cap T$, we have

$$|P \cap (C \cup T)| = |P \cap C| + |P \cap T| - |P \cap C \cap T| = 8 + 7 - 4 = 11.$$

So $|P \setminus (C \cup T)| = 15 - 11 = 4$, that is, there are 4 plots on which only potatoes are grown.

c) Applying the inclusion-exclusion principle to the two sets $P \cap C = C \cap P$ and $C \cap T$, we have

$$|C \cap (P \cup T)| = |C \cap P| + |C \cap T| - |C \cap P \cap T| = 8 + 6 - 4 = 10.$$

So $|C \setminus (P \cup T) = 12 - 10 = 2$, that is, there are two plots on which only courgettes are grown.

d) Applying the inclusion-exclusion principle to the two sets $T \cap P = P \cap T$ and $C \cap T = T \cap C$, we have

$$|T \cap (P \cup C)| = |T \cap P| + |T \cap C| - |T \cap P \cap C| = 7 + 6 - 4 = 9.$$

So $|T \setminus (P \cup C) = 10 - 9 = 1$, that is, there is one plot on which only tomatoes are grown.

7.

a) If $y \in f(f^{-1}(B))$ then $y = f(x)$ for x such that $f(x) \in B$, that is, $y \in B$. So $f(f^{-1}(B)) \subset B$

b) If $x \in A$ then $f(x) \in f(A)$ and hence $x \in f^{-1}(f(A))$, that is $A \subset f^{-1}(f(A))$.

If B is not contained in $\text{Im}(f)$ then B cannot be equal to $f(f^{-1}(B))$. For example if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ and $B = \mathbb{R}$ then $f(f^{-1}(B)) = f(\mathbb{R}) = [0, \infty)$. Using this same f we can produce a set A such that $A \neq f^{-1}(f(A)$. If $f(x_1) = f(x_2)$ and $x_1 \in A$ then $x_2 \in f^{-1}(f(A))$. So if we take $A = [0.\infty)$ we have $f^{-1}(f(A)) = f^{-1}([0, \infty)) = \mathbb{R}$.

8. **Base case** We have $x_0 = 1$ and $1^2 < 2$, so $x_n \geq 1$ and $x_n^2 < 2$ are true for $n = 0$.

Inductive step Now suppose that $x_n \geq 1$ and $x_n^2 < 3$. Then $x_n \geq 1 \Rightarrow 3x_n + 2 \geq x_n + 4 > x_n + 3 > 0$ and hence $x_{n+1} = \dfrac{3x_n + 2}{x_n + 3} > 1$.

Also

$$x_{n+1}^2 - 2 = \left(\frac{3x_n + 2}{x_n + 3}\right)^2 = \frac{9x_n^2 + 12x_n + 4 - 2x_n^2 - 12x_n - 18}{(x_n + 3)^2} = \frac{7x_n^2 - 14}{(x_n + 3)^2} =< 0$$

So

$$(1 \leq x_n) \wedge (x_n^2 < 2) \Rightarrow (1 < x_{n+1}) \wedge (x_{n+1}^2 < 2)$$

So by induction $1 \leq x_n$ and $x_n^2 < 2$ for all $n \in \mathbb{N}$.

9.

a)

$$A = (A \setminus (B \cup C)) \cup (A \cap B \setminus C) \cup (A \cap C \setminus B) \cup (A \cap B \cap C).$$

$$B = (B \setminus (A \cup C)) \cup (B \cap A \setminus C) \cup (B \cap C \setminus A) \cup (A \cap B \cap C).$$

$$C = (C \setminus (A \cup B)) \cup (C \cap A \setminus B) \cup (B \cap C \setminus A) \cup (A \cap B \cap C).$$

$$A \cup B \cup C = (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \cup (A \cap B \setminus C) \cup (A \cap C \setminus B) \cup (B \cap C \setminus A) \cup (A \cap B \cap C).$$

b) From these we obtain, since all the sets in each union are disjoint,

$$|A| = 75 = |A \setminus (B \cup C)| + |A \cap B \setminus C| + |A \cap C \setminus B| + |A \cap B \cap C| \quad (1)$$

$$|B| = 60 = |B \setminus (A \cup C)| + |A \cap B \setminus C| + |B \cap C \setminus A| + |A \cap B \cap C| \quad (2)$$

$$|C| = 45 = |C \setminus (A \cup B)| + |B \cap C \setminus C| + |A \cap C \setminus B| + |A \cap B \cap C| \quad (3)$$

$$100 = |A \cup B \cup C| = \begin{array}{l} |A \setminus (B \cup C))| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ +|A \cap B \setminus C| + |A \cap C \setminus B| + |B \cap C \setminus A| + |A \cap B \cap C| \end{array} \quad (4)$$

Adding equations (1), (2) and (3) and then subtracting (4), we obtain

$$80 = |A \cap B \setminus C| + |A \cap C \setminus B| + |B \cap C \setminus A| + 2|A \cap B \cap C| \quad (5)$$

From adding (1), (2) and (3) and rearranging, and using (5) we obtain

$$\begin{array}{l} |A \setminus (B \cup C)| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ = 180 - 2(|A \cap B \setminus C| + |A \cap C \setminus B| + |B \cap C \setminus A| + 2|A \cap B \cap C|) + |A \cap B \cap C| \\ = 20 + |A \cap B \cap C| \end{array} \quad (6)$$

The first line is the number of people speaking exactly one of English, Spanish or Swahili. From the third line we see that maximising this number is the same as maximising $|A \cap B \cap C|$. But from (5) we see that $|A \cap B \cap C| \leq 40$ and that if $|A \cap B \cap C| = 40$ then

$$(A \cap B \setminus C) = (A \cap C \setminus B) = (B \cap C \setminus A) = (A \cap B \cap C) = \emptyset.$$

From (1) (2) and (3) we then obtain

$$|A \cap B \cap C| = 75 - 40 = 35$$

$$|B \setminus (A \cup C)| = 60 - 40 = 20,$$

$$|C \setminus (A \cup B)| = 45 = 40 = 5$$

So the number of people speaking just English is 35, the number speaking just Spanish is 20 and the number speaking just Swahili is 5. Altogether the number speaking just one language is 35+20+5 = 60.

c)

$$75 = |A| = |A \setminus (B \cup C)| + |A \cap (B \cup C)|,$$

$$B \cup C| = |(B \cup C) \setminus A| + |A \cap (B \cup C)|,$$

$$100 = |A \cup B \cup C| = |A \setminus (B \cup C)| + |(B \cup C) \setminus A| + |A \cap (B \cup C)| = 75 + |(B \cup C) \setminus A|$$

So

$$B \cup C| = 25 + |A \cap (B \cup C)|,$$

So to maximise $|A \setminus (B \cup C)|$ we have to minimize $|B \cup C|$. But $|B \cup C|$ is minimised when $C \subset B$, that is, everyone who speaks Swahili also speaks Spanish, which means that

$$45|C| = |B \cap C|$$

$$60 = |B| = |B \cup C|$$

So nobody speaks just Swahili, 15 speak just Spanish, 15 speak English and one other language, and 60 speak just English.