MATH105 Feedback and Solutions 7

1.

a) There is no inverse function because f is not a bijection: in fact f is neither injective nor surjective.

b) $f^{-1} : [0, \infty) \to (-\infty, 0]$ is defined by $f^{-1}(x) = -\sqrt{x}$

By convention, $\sqrt{x}$ means $+\sqrt{x}$. The codomain of f^{-1} is $(-\infty, 0]$, so $f^{-1}(x) = -\sqrt{x}$.

c)

$$y = \frac{1}{x+2} \Leftrightarrow yx + 2y = 1 \Leftrightarrow xy = 1 - 2y \Leftrightarrow x = \frac{1-2y}{y}$$

So $f^{-1}(x) = \dfrac{1-2x}{x}$.

d)

$$2e^x = y \Leftrightarrow x = \ln(y/2).$$

So $f^{-1}(x) = \ln(x/2) = \ln x - \ln 2$.

Marks were deducted if the inverse was not given explicitly,that is, if there was no statement such as "$f^{-1}(x) = \ln(x/2)$". Marks were also deducted for statements such as "$f^{-1}(x) = \ln(y/2)$". Both "$f^{-1}(x) = \ln(x/2)$" and "$f^{-1}(y) = \ln(y/2)$" are correct, however.

2. Let C, D and H be the sets of children owning cats, dogs and hamsters respectively. Then

$$|C| = |D| = 15, \quad |H| = 10, \quad |C \cup D \cup H| = 22,$$

$$|C \cap D| = 8, \; |C \cap H| = 6, \; |D \cap H| = 5.$$

a) By the Inclusion-Exclusion Principle for 3 sets,

$$22 = 15 + 15 + 10 - 8 - 6 - 5 + |C \cap D \cap H|$$

So the number of children in $C \cap D \cap H$, that is, with all three pets, is $22 - 40 + 19 = 1$

b) This means that, using the Inclusion-Exclusion Principle for 2 sets,

$$|D \cap (C \cup H)| = |(D \cap C) \cup (D \cap H)| = |D \cap C| + |D \cap H| - |D \cap C \cap H| = 5 + 8 - 1 = 12,$$

using $(D \cap C) \cap (D \cap H) = D \cap C \cap H$. So the number of children with just a dog is $15 - 12 = 3$.

Alternatively, since all the pets are dogs, cats or hamsters, the number of children with just a dog is

$$|D \cup C \cup H| - |C \cup H| = |D \cup C \cup H| - (|C| + |H| - |C \cap H|) = 22 - (15 + 10 - 6) = 22 - 25 + 6 = 3.$$

This method also uses the Inclusion-Exclusion Principle for 2 sets.

c) Similarly

$$|C \cap (H \cup D)| = |C \cap H| + |C \cap D| - |C \cap H \cap D| = 6 + 8 - 1 = 13$$

So the number of children with a cat and one other pet is 13 and the number with just a cat is $15 - 13 = 2$.

Alternatively, the number of children with just a cat is

$$|C \cup D \cup H| - |C \cup H| = |C \cup D \cup H| - (|D| + |H| - |D \cap H|) = 22 - (15 + 10 - 5) = 22 - 25 + 5 = 2.$$

d) Similarly

$$|H \cap (C \cup D)| = |H \cap C| + |H \cap D| - |H \cap C \cap D| = 6 + 5 - 1 = 10$$

So every child with a hamster also has either a cat of a dog and no child just has a hamster.

Alternatively, the number of children with just a hamster is

$$|C \cup D \cup H| - |C \cup D| = |C \cup D \cup H| - (|C| + |D| - |C \cap D|) = 22 - (15 + 15 - 8) = 22 - 30 + 8 = 2.$$

All the solutions shown here use the Inclusion-Exclusion Principle for two or three sets. I did require some explanation of how formulas were derived for full marks, at least for the first calculation. It is possible to simply draw the Venn diagram and solve the linear equations, and I saw a number of solutions that used a Venn diagram. I accepted this if the diagram was clear and if it was clear what the numbers represented in each region. Be careful to distinguish between a set like H (in this example, the set of children with hamsters) and the number of elements in H, which is denoted by $|H|$

3.

a)

$$x \in f^{-1}(B \cup C) \Leftrightarrow f(x) \in B \cup C \Leftrightarrow fIx) \in B \vee f(x) \in C \Leftrightarrow x \in f^{-1}(B) \cup f^{-1}(C).$$

So $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$

b)

$$x \in f^{-1}(B \cap C) \Leftrightarrow f(x) \in B \cap C \Leftrightarrow f(x) \in B \wedge f(x) \in C \Leftrightarrow x \in f^{-1}(B) \cap f^{-1}(C).$$

So $f^{-1}(B \cup C) = f^{-1}(B) \cap f^{-1}(C)$

It is only when f is a bijection that the inverse function f^{-1} exists and that $f^{-1}(B) = \{f^{-1}(y) : y \in B\}$. The general definition $f^{-1}(B) = \{x \in X : f(x) \in B\}$ works for any function $f : X \to Y$. Be careful to distinguish between $\cup$ and $\vee$, and between $\cap$ and $\wedge$. In both cases these pairs of symbols are related, but they are used in slightly different ways. For example $\cup$ means "the union of"(for sets) and $\vee$ means "or". So the set $B \cup C$, for example , is the set of elements which are in B or C (possibly both), that is, $B \cup C = \{x : x \in B \vee x \in C\}$

4. Base case We have $x_0 = 1$ and $1^2 < 3$, so $x_n \geq 1$ and $x_n^2 < 3$ are true for $n = 0$.

Inductive step Now suppose that $x_n \geq 1$ and $x_n^2 < 3$. Then $x_n \geq 1 \Rightarrow 2x_n + 3 \geq x_n + 5 > x_n + 2\,0$ and hence $x_{n+1} = \dfrac{2x_n + 3}{x_n + 2} \geq> 1$.

Also

$$x_{n+1}^2 - 3 = \left(\frac{2x_n + 3}{x_n + 2}\right)^2 - 3 = \frac{4x_n^2 + 12x_n + 9 - 3x_n^2 - 12x_n - 12}{(x_n + 2)} = \frac{x_n^2 - 3}{(x_n + 2)^2} < 0$$

So

$$(1 \leq x_n) \wedge (x_n^2 < 3) \Rightarrow (1 < x_{n+1}) \wedge (x_{n+1}^2 < 3)$$

So by induction $1 \leq x_n$ and $x_n^2 < 3$ for all $n \in \mathbb{N}$.

There are still some misconceptions on Induction in some quarters but I hope they are decreasing. Perserverance is what counts. Don't give up, and you will get there. This was not the easiest example.

- *The natural base case to take in this example is $n = 0$ since $1 \leq 1 = x_0$ and $1 = x_0^2 < 3$. It is not wrong to take $n = 1$ or 2 as base but it is more work and it is unnecessary.*
- *I still saw some backwards working. the inductive step is* assume *that $1 \leq x_n$ and* deduce *that $1 \leq x_{n+1}$, not the other way round.*
- *I saw some confusion about notation that I could not quite figure out but it appeared that some people find inductive definitions difficult. I saw some cases where people were working with x_{n+2} instead of x_{n+1}. There were also one or two where x_n was confused with n.*

- *In order to deduce from* $1 \le x_n$ *that* $1 \le \dfrac{2x_n+3}{x_n+2}$ *it is simplest to use* $0 < x_n + 2 \le 2x_n + 3$ *– which is of course true, even if we just have* $x_n \ge 0$ *(or even* $x_n \ge -1$*). It is possible to use the fact that* $\dfrac{2x+3}{x+2}$ *is an increasing function and hence* $x_n \ge 1 \Rightarrow x_{n+1} \ge (2+3)/(1+2) = 5/3$*. But I only accepted this method if it was checked that* $\dfrac{2x+3}{x+2}$ *is an increasing function, for example by showing that*

$$\frac{2x+3}{x+2} = 2 - \frac{1}{x+2}$$

and noting that $\dfrac{1}{x+2}$ *is decreasing.*

- *In part b), to show that* $x_{n+1}^2 < 3$*, the formula for* x_{n+1} *is*

$$x_{n+1}^2 = \frac{(2x_n+3)^2}{(x_n+2)^2} = \frac{4x_n^2 + 12x_n + 9}{x_n^2 + 4x_n + 4}$$

Don't forget the cross terms!

I think you may have seen similar examples in MATH101, and $\lim_{n\to\infty} x_n = \sqrt{3}$*. The equation*

$$x = \frac{2x+3}{x+2}$$

has solutions $x = \pm\sqrt{3}$ *and we know that* $x_n \ge 1$ *for all* n*. But this was not part of the question.*

Solutions to Practice Problems

5.

a) The inverse does not exist, because f is not surjective. In fact, since (as shown in lectures)

$$1 + x + x^2 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\mathrm{Im}(f) = [3/4, \infty)$.

b) This time

$$y = 1 + x + x^2 \quad \Leftrightarrow \quad y - \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 \quad \Leftrightarrow \quad x = -\frac{1}{2} + \sqrt{y - \frac{3}{4}}$$

So $f^{-1} : [3/4, \infty) \to [-1/2, \infty)$ is defined by

$$f^{-1}(y) = -\frac{1}{2} + \sqrt{y - \frac{3}{4}}.$$

c) We have

$$y = \frac{1}{x+1} \quad \Leftrightarrow \quad xy + y = 1 \quad \Leftrightarrow \quad x = \frac{1}{y} - 1$$

from which we see that $x > -1 \Leftrightarrow y > 0$. So $f^{-1} : (0, \infty) \to (-1, \infty)$ does exist and is defined by

$$f^{-1}(y) = \frac{1}{y} - 1.$$

6. Let P, C and T denote the sets of plots growing, potatoes, courgettes and tomatoes respectively.

a) The inclusion-exclusion principle for three sets says that

$$|P \cup C \cup T| = |P| + |C| + |T| - |P \cap C| - |P \cap T| - |C \cap T| + |P \cap C \cap T|,$$

that is

$$20 = 15 + 12 + 10 - 8 - 7 - 6 + |P \cap C \cap T|$$

that is $|P \cap C \cap T| = 4$, that is, 4 plots grow all 3 crops.

b) The set of plots growing only potatoes is $P \setminus (C \cup T)$ and

$$|P \setminus (C \cup T)| = |P| - |P \cap (C \cup T)|.$$

Applying the inclusion-exclusion principle for two sets to the sets $P \cap C$ and $P \cap T$, we have

$$|P \cap (C \cup T)| = |P \cap C| + |P \cap T| - |P \cap C \cap T| = 8 + 7 - 4 = 11.$$

So $|P \setminus (C \cup T)| = 15 - 11 = 4$, that is, there are 4 plots on which only potatoes are grown.

c) Applying the inclusion-exclusion principle to the two sets $P \cap C = C \cap P$ and $C \cap T$, we have

$$|C \cap (P \cup T)| = |C \cap P| + |C \cap T| - |C \cap P \cap T| = 8 + 6 - 4 = 10.$$

So $|C \setminus (P \cup T) = 12 - 10 = 2$, that is, there are two plots on which only courgettes are grown.

d) Applying the inclusion-exclusion principle to the two sets $T \cap P = P \cap T$ and $C \cap T = T \cap C$, we have

$$|T \cap (P \cup C)| = |T \cap P| + |T \cap C| - |T \cap P \cap C| = 7 + 6 - 4 = 9.$$

So $|T \setminus (P \cup C) = 10 - 9 = 1$, that is, there is one plot on which only tomatoes are grown.

7.

a) If $y \in f(f^{-1}(B))$ then $y = f(x)$ for x such that $f(x) \in B$, that is, $y \in B$. So $f(f^{-1}(B)) \subset B$

b) If $x \in A$ then $f(x) \in f(A)$ and hence $x \in f^{-1}(f(A))$, that is $A \subset f^{-1}(f(A))$.

If B is not contained in $\text{Im}(f)$ then B cannot be equal to $f(f^{-1}(B))$. For example if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ and $B = \mathbb{R}$ then $f(f^{-1}(B)) = f(\mathbb{R}) = [0, \infty)$. Using this same f we can produce a set A such that $A \neq f^{-1}(f(A)$. If $f(x_1) = f(x_2)$ and $x_1 \in A$ then $x_2 \in f^{-1}(f(A))$. So if we take $A = [0.\infty)$ we have $f^{-1}(f(A)) = f^{-1}([0, \infty)) = \mathbb{R}$.

8. Base case We have $x_0 = 1$ and $1^2 < 2$, so $x_n \geq 1$ and $x_n^2 < 2$ are true for $n = 0$.

Inductive step Now suppose that $x_n \geq 1$ and $x_n^2 < 3$. Then $x_n \geq 1 \Rightarrow 3x_n + 2 \geq x_n + 4 > x_n + 3 > 0$ and hence $x_{n+1} = \dfrac{3x_n + 2}{x_n + 3} > 1$.

Also

$$x_{n+1}^2 - 2 = \left(\frac{3x_n + 2}{x_n + 3}\right)^2 = \frac{9x_n^2 + 12x_n + 4 - 2x_n^2 - 12x_n - 18}{(x_n + 3)^2} = \frac{7x_n^2 - 14}{(x_n + 3)^2} =< 0$$

So

$$(1 \leq x_n) \wedge (x_n^2 < 2) \Rightarrow (1 < x_{n+1}) \wedge (x_{n+1}^2 < 2)$$

So by induction $1 \leq x_n$ and $x_n^2 < 2$ for all $n \in \mathbb{N}$.

Extra question

9.

a)

$$A = (A \setminus (B \cup C)) \cup (A \cap B \setminus C) \cup (A \cap C \setminus B) \cup (A \cap B \cap C).$$
$$B = (B \setminus (A \cup C)) \cup (B \cap A \setminus C) \cup (B \cap C \setminus A) \cup (A \cap B \cap C).$$
$$C = (C \setminus (A \cup B)) \cup (C \cap A \setminus B) \cup (B \cap C \setminus A) \cup (A \cap B \cap C).$$
$$A \cup B \cup C = (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \cup (A \cap B \setminus C) \cup (A \cap C \setminus B) \cup (B \cap C \setminus A) \cup (A \cap B \cap C).$$

b) From these we obtain, since all the sets in each union are disjoint,

$$|A| = 75 = |A \setminus (B \cup C)| + |A \cap B \setminus C| + |A \cap C \setminus B| + |A \cap B \cap C| \quad (1)$$
$$|B| = 60 = |B \setminus (A \cup C)| + |A \cap B \setminus C| + |B \cap C \setminus A| + |A \cap B \cap C| \quad (2)$$
$$|C| = 45 = |C \setminus (A \cup B)| + |B \cap C \setminus C| + |A \cap C \setminus B| + |A \cap B \cap C| \quad (3)$$
$$100 = |A \cup B \cup C| = \begin{array}{l} |A \setminus (B \cup C))| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ +|A \cap B \setminus C| + |A \cap C \setminus B| + |B \cap C \setminus A| + |A \cap B \cap C| \end{array} \quad (4)$$

Adding equations (1), (2) and (3) and then subtracting (4), we obtain

$$80 = |A \cap B \setminus C| + |A \cap C \setminus B| + |B \cap C \setminus A| + 2|A \cap B \cap C| \quad (5)$$

From adding (1), (2) and (3) and rearranging, and using (5) we obtain

$$\begin{array}{l} |A \setminus (B \cup C)| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| \\ = 180 - 2(|A \cap B \setminus C| + |A \cap C \setminus B| + |B \cap C \setminus A| + 2|A \cap B \cap C|) + |A \cap B \cap C| \\ = 20 + |A \cap B \cap C| \end{array} \quad (6)$$

The first line is the number of people speaking exactly one of English, Spanish or Swahili. From the third line we see that maximising this number is the same as maximising $|A \cap B \cap C|$. But from (5) we see that $|A \cap B \cap C| \leq 40$ and that if $|A \cap B \cap C| = 40$ then

$$(A \cap B \setminus C) = (A \cap C \setminus B) = (B \cap C \setminus A) = (A \cap B \cap C) = \emptyset.$$

From (1) (2) and (3) we then obtain

$$|A \cap B \cap C| = 75 - 40 = 35$$
$$|B \setminus (A \cup C)| = 60 - 40 = 20,$$
$$|C \setminus (A \cup B)| = 45 = 40 = 5$$

So the number of people speaking just English is 35, the number speaking just Spanish is 20 and the number speaking just Swahili is 5. Altogether the number speaking just one language is 35+20+5 = 60.

c)

$$75 = |A| = |A \setminus (B \cup C)| + |A \cap (B \cup C)|,$$
$$B \cup C| = |(B \cup C) \setminus A| + |A \cap (B \cup C)|,$$
$$100 = |A \cup B \cup C| = |A \setminus (B \cup C)| + |(B \cup C) \setminus A| + |A \cap (B \cup C)| = 75 + |(B \cup C) \setminus A|$$

So

$$B \cup C| = 25 + |A \cap (B \cup C)|,$$

So to maximise $|A \setminus (B \cup C)|$ we have to minimize $|B \cup C|$. But $|B \cup C|$ is minimised when $C \subset B$, that is, everyone who speaks Swahili also speaks Spanish, which means that

$$45|C| = |B \cap C|$$
$$60 = |B| = |B \cup C|$$

So nobody speaks just Swahili, 15 speak just Spanish, 15 speak English and one other language, and 60 speak just English.