MATH105 Feedback and Solutions 5

1. Base case $2^2 + 3^2 = 4 + 9 = 13 < 16 = 4^2$. So $2^n + 3^n < 4^n$ is true for $n = 2$.
Inductive step Now assume that $n \geq 2$ and $2^n + 3^n < 4^n$ and consider $2^{n+1} + 3^{n+1}$. We have

$$2^{n+1} + 3^{n+1} < 3 \cdot (2^n + 3^n) < 3 \cdot 4^n < 4^{n+1}$$

So

$$2^n + 3^n < 4^n \Rightarrow 2^{n+1} + 3^{n+1} < 4^{n+1}$$

Finishing By induction $2^n + 3^n < 4^n$ for all integers $n \geq 2$.

In quite a lot of homeworks that I saw, either the inductive step or the "Finishing off" was not written properly. The inductive step is to prove "if true for n then true for $n+1$". A common omission was to leave out "if true for n", or sometimes to only put it later on. It should come at the start. Note also that the "finishing off" is "so true for all $n \geq 2$" (or a variation of that if the base case is different). The finishing off statement refers to all n not *all $n + 1$.*

Some people are writing $n = k$ and $n = k + 1$. That is fine. But if you are going to do this, and prove that a statement for $n = k$ implies it for $n = k + 1$, please do not switch to $n + 1$

2.

$$\left.\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 434 \\ 364 \end{matrix} \quad \begin{matrix} R_1 - R_2 \\ \rightarrow \end{matrix} \quad \left.\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 70 \\ 364 \end{matrix} \quad \begin{matrix} \rightarrow \\ R_2 - 5R_1 \end{matrix} \quad \left.\begin{matrix} 1 & -1 \\ -5 & 6 \end{matrix}\right| \begin{matrix} 70 \\ 14 \end{matrix}$$

$$\begin{matrix} R_1 - 5R_2 \\ \rightarrow \end{matrix} \quad \left.\begin{matrix} 26 & -31 \\ -5 & 6 \end{matrix}\right| \begin{matrix} 0 \\ 14 \end{matrix}$$

So the g.c.d. is $d = 14$, from the second row of the last matrix. Also from this second row we have

$$-5 \times 434 + 6 \times 364 = 14$$

so $a = -5$ and $b = 6$ The first row of the last matrix gives $26 \times 434 = 31 \times 364$. This number is $11,284$ and is the l.c.m..

Using prime factorisation, we have $364 = 4 \times 91 = 2^2 \times 7 \times 13$ and $434 = 2 \times 217 = 2 \times 7 \times 31$. So the g.c.d. is 14 and the l.c.m. is $2^2 \times 7 \times 13 \times 31 = 364 \times 31 = 11,284$.

3. $p_5 = 11$, $p_6 = 13$ and $p_7 = 17$. So

1. $(p_5, p_6) \cap \mathbb{Z}$=$\{$ 12$\}$;
2. $(p_6, p_7) \cap \mathbb{Z} = \{14, 15, 16\}$.

Note the use of curly brackets. I have written them in on some scripts. They should be used if you are writing in terms of sets, as is don in the solution above $\{a\}$ means "the set containing a".

4. If $n \in (1327, 1361) \cap \mathbb{Z}$ then $n = k\ell$ for some $k,\ \ell \in \mathbb{Z}_+$ with $2 \leq k \leq \ell \in \mathbb{Z}_+$. Then $k^2 \leq l\ell = n \leq 1360 < 1369 = 37^2$ and so $k < 37$. Then $k_1 \leq k < 37$ for any prime divisor k_1 of k, and k_1 is also a divisor of n. Since k_1 is prime, $k_1 \leq 31$.

Some sort of explanation was required for full marks: something more than just computing 37^2 and 31^2.

5. The only positive divisors of 1327 are 1 and 1327, because 1327 is prime. But 1327 is not a divisor of n, since $1 \leq n < 1327$. So the only possible positive divisor of both n and 1327 is 1, and this is the g.c.d..

a) Now using the Euclidean algorithm to find a and $b \in \mathbb{Z}$ such that $1327a + 17b = 1$:

$$\left.\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 1327 \\ 17 \end{matrix} \quad \begin{matrix} R_1 - 78R_2 \\ \rightarrow \end{matrix} \quad \left.\begin{matrix} 1 & -78 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 1 \\ 17 \end{matrix}$$

which gives $a = 1$ and $b = -78$, that is $1 \times 1327 - 78 \times 17 = 1$.

b) Using the Euclidean algorithm to find a and $b \in \mathbb{Z}$ such that $1327a + 31b = 1$:

$$\left.\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 1327 \\ 31 \end{matrix} \quad \begin{matrix} R_1 - 42R_2 \\ \rightarrow \end{matrix} \quad \left.\begin{matrix} 1 & -42 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 25 \\ 31 \end{matrix} \quad \begin{matrix} \rightarrow \\ R_2 - R_1 \end{matrix} \quad \left.\begin{matrix} 1 & -42 \\ -1 & 43 \end{matrix}\right| \begin{matrix} 25 \\ 6 \end{matrix}$$

$$\begin{array}{ccc} R_1 - 4R_2 & 5 & -214 \,|\, 1 \\ \rightarrow & -1 & 43 \,|\, 6 \end{array}$$

which gives $a = 5$ and $b = -214$, that is $5 \times 1327 - 214 \times 31 = 1$.

6. Using the first rows of the second matrices in a) and b) for each of the first two parts:

a) $1327 = 78 \times 17 + 1$;

b) $1327 = 42 \times 31 + 25$;

c) $1327 = 102 \times 13 + 1$;

d) $1327 = 69 \times 19 + 16$;

e) $1327 = 57 \times 23 + 16$;

f) $1327 = 45 \times 29 + 22$.

7. From question 5, we can see which numbers in the prime gap are divisible by each of 13, 17, 19, 23, 29 and 31. In fact we have

$$1330 = 70 \times 19 = 2 \times 5 \times 7 \times 19,\ 1333 = 43 \times 31,\ 1334 = 58 \times 23 = 2 \times 23 \times 29,\ 1339 = 103 \times 13, 1343 = 79 \times 17,$$

$$1349 = 71 \times 19,\ 1352 = 104 \times 13 = 2^3 \times 13^2, 1357 = 59 \times 23,\ 1360 = 80 \times 17 = 2^4 \times 5 \times 17.$$

This was not part of the question but *just for the record* here are the prime factorisations of all the numbers in the gap. Numbers divisible by 11 are easy to spot:

$$1331 = 11^3,\ 1342 = 11 \times 122 = 2 \times 11 \times 61,\ 1353 = 11 \times 123 = 3 \times 11 \times 41$$

We have seen that 1330 is divisible by 7. So the others are

$$1337 = 7 \times 191,\ 1344 = 2^6 \times 3 \times 7,\ 1351 = 7 \times 193, 1358 = 2 \times 7 \times 97.$$

Of the numbers divisible by 5, we have already dealt with 1330 and 1360. For the others, we have

$$1335 = 5 \times 267 = 3 \times 5 \times 89,\ 1340 = 2^2 \times 5 \times 67,\ 1345 = 5 \times 269,\ 1350 = 5^2 \times 54 = 2 \times 3^3 \times 5^2,$$

$$1355 = 5 \times 271.$$

The numbers in the gap which are divisible by 3 start with $1329 = 3 \times 443$ and end with $1359 = 3 \times 453 = 3^2 \times 151$. The prime numbers in this range are 443 and 449. The others must already have been factorised, apart from those divisible by 2. The numbers divisible by 2 in the gap start with $1328 = 2 \times 664 = 2^4 \times 83$ and end with $1360 = 2 \times 680 = 2^4 \times 85 = 2^4 \times 5 \times 17$. The prime numbers between 664 and 680 are 673 and 677, where $673 \times 2 = 1346$ and $677 \times 2 = 1354$.

So the complete list of prime factorisations (for the record) is

$$1328 = 2^4 \times 83,\ 1329 = 3 \times 443,\ 1330 = 2 \times 5 \times 7 \times 19,\ 1331 = 11^3,\ 1332 = 2^2 \times 3 \times 37,\ 1333 = 31 \times 43,$$

$$1334 = 2 \times 23 \times 29,\ 1335 = 3 \times 5 \times 89,\ 1336 = 2^2 \times 3 \times 113,\ 1337 = 7 \times 191,\ 1338 = 2 \times 3 \times 223,\ 1339 = 13 \times 103,$$

$$1340 = 2^2 \times 5 \times 67,\ 1341 = 3^2 \times 149,\ 1342 = 2 \times 11 \times 61,\ 1343 = 17 \times 79,\ 1344 = 2^6 \times 3 \times 7,$$

$$1345 = 5 \times 269,\ 1346 = 2 \times 673,\ 1347 = 3 \times 449,\ 1348 = 2^2 \times 337,\ 1349 = 19 \times 71,\ 1350 = 2 \times 3^3 \times 5^2,$$

$$1351 = 7 \times 193,\ 1352 = 2^3 \times 167,\ 1353 = 3 \times 11 \times 41,\ 1354 = 2 \times 677,\ 1355 = 5 \times 271,\ 1356 = 2^3 \times 167,$$

$$1357 = 23 \times 59,\ 1358 = 2 \times 7 \times 97,\ 1359 = 3^2 \times 151,\ 1360 = 2^4 \times 5 \times 17.$$

Solutions to Practice Problems

8. Base case $3^3 + 4^3 = 27 + 64 = 91 < 125 = 5^3$. So $3^n + 4^n < 5^n$ is true for $n = 3$.
Inductive step Now assume that $n \geq 3$ and $3^n + 4^n < 5^n$ and consider $3^{n+1} + 4^{n+1}$. We have

$$3^{n+1} + 4^{n+1} < 4 \cdot (3^n + 4^n) < 4 \cdot 5^n < 5^{n+1}$$

So

$$3^n + 4^n < 5^n \Rightarrow 3^{n+1} + 4^{n+1} < 5^{n+1}$$

Finishing By induction $3^n + 4^n < 5^n$ for all integers $n \geq 3$.

9.

$$\begin{array}{cc|c} 1 & 0 & 450 \\ 0 & 1 & 378 \end{array} \quad \begin{array}{c} R_1 - R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 1 & -1 & 72 \\ 0 & 1 & 378 \end{array} \quad \begin{array}{c} \to \\ R_2 - 5R_1 \end{array} \quad \begin{array}{cc|c} 1 & -1 & 72 \\ -5 & 6 & 18 \end{array}$$

$$\begin{array}{c} R_1 - 4R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 21 & -25 & 0 \\ -5 & 6 & 18 \end{array}$$

So the g.c.d. is $d = 18$, from the second row of the last matrix. Also from this second row we have

$$-5 \times 434 + 6 \times 364 = 14$$

so $a = -5$ and $b = 6$ The first row of the last matrix gives $21 \times 450 = 25 \times 378$. This number is 9450, and is the l.c.m..

10. We fix $n \in \mathbb{Z}_+$. Since $k \mid n! = \prod_{j=1}^{n} j$ for $2 \leq k \leq n$, it is also true that $k \mid n! + k$ for $2 \leq k \leq n$. Since k is divisible by at least one prime for any integer $k \geq 2$, and $n! + k > k$, it follows that $n! + k$ is not prime for any $2 \leq k \leq n$. There are $n - 1$ such numbers and hence they must all be contained in the same prime gap, which must have length $\geq n$.