MATH105 Feedback and Solutions 3

1. Base case We have $a_0 = 3 = 2^1 + 1$. So $a_n = 2^{n+1} + 1$ is true for $n = 0$.
Inductive step Now suppose that $a_n = 2^{n+1} + 1$ for some $n \in \mathbb{N}$. Then

$$a_{n+1} = 2a_n - 1 = 2(2^{n+1} + 1) - 1 = 2^{n+2} + 2 - 1 = 2^{n+2} + 1$$

So

$$a_n = 2^{n+1} + 1 \Rightarrow a_{n+1} = 2^{n+2} + 1$$

So by induction $a_n = 2^{n+1} + 1$ for all $n \in \mathbb{N}$.

The inductive step is to assume *a statement for n and* prove *a statement for $n+1$. In this question – which was probably found the easiest by most people – there was also an* inductive definition *of a_{n+1} in terms of a_n, which was* given, *and which was $a_{n+1} = 2a_n - 1$. Most answers that I saw understand what was to be proved and what was given. The inductive definition $a_{n+1} = 2a_n - 1$ was* given *and the formula $a_n = 2^{n+1} + 1$ was to be* proved, *by induction, for all $n \in \mathbb{N}$. Most answers I saw did prove the formula, or made a good attempt to prove it, but there were a few answers which assumed the formula and then deduced the inductive definition from it.*

Some people are writing the inductive step as " true for $n = k$, implies for $n = k+1$ ...hence true for all $n \geq 0$ (or 1 or 7) by induction." This is fine, but is optional. I am not promoting it strongly because there are more advanced induction questions in which induction is on two different integers, for example. Also, do be careful *not to mix up n and k in the same formula. Do* not *write, for example "$a_{n+1} = 2(2^{k+1} + 1) - 1$"*

Here is a repeat of the basic procedure in induction (See also sheet 2.)

Base case: prove "it" (whatever the statement is) for n_0 (whatever the first integer is)

Inductive step: Assume "it" is true *for n (or k) where n (or k) is any fixed integer $\geq n_0$ and from this assumption* prove "it" *is true for $n+1$ (or $k+1$)*

Hence by induction "it" must be true for all $n \in \mathbb{N}$ with $n \geq n_0$.

This last "finishing off " step was missing in some answers I saw. I am only giving full marks when I can see something that I can recognise as "finishing off" - even though that is only one mark.

Induction works because of the nature of the set of integers. If a set includes an integer n_0 – the base case — and includes $n+1$ whenever if includes n, then it includes all integers $\geq n_0$.

2. We have

$$a_1 = 1,$$
$$a_{n+1} = \frac{6a_n + 5}{a_n + 2}, \ k \in \mathbb{Z}_+.$$

(i) So $a_1 > 0$ (This is the *base case.* If $a_n > 0$ then $a_n + 2 > 0$ and $6a_n + 5 > 0$ and hence $a_{n+1} > 0$ (This is the *inductive step.* So by induction $a_n > 0$ for all $n \geq 1$.

(ii) Clearly $a_1 < 5$ (This is the *base case.*) Now for the *inductive step*: assume inductively that $0 < a_n < 5$. Then $0 < a_n + 2$ and

$$a_{n+1} = \frac{6a_n + 5}{a_n + 2} < \frac{5a_n + 10}{a_n + 2} = 5.$$

So $0 < a_n < 5 \Rightarrow 0 < a_{n+1} < 5$ and by induction $0 < a_n < 5$ for all $n \geq 1$

Some people took the base case in this question as $n = 2$ – often without realising it. I think it must have been because the base case $n = 1$ was so easy. The base case held because $a_1 = 1$ satisfies $0 < 1 < 5$. I also saw a number of probably unintentional variants of the inductive step. One was "if true for $n+1$ then true for $n+2$". It is permissible to prove "True for $n+1$ if and only if true for n" in order to prove "if true for n then true for $n+1$" but if this is done then the "if and only if" symbol $\Leftrightarrow$ should be used. *For example: "Suppose that $a_n > 0$. Then*

$$a_{n+1} < 5 \Leftrightarrow \frac{6a_n + 5}{a_n + 2} < 5 \Leftrightarrow 6a_n + 5 < 5a_n + 10 \Leftrightarrow a_n < 5$$

Hence $0 < a_n < 5 \Rightarrow a_{n+1} < 5$" The assumption that $a_n > 0$ – or at least $a_n + 2 > 0$ — is needed in order to pass from $\frac{6a_n + 5}{a_n + 2} < 5$ to $6a_n + 5 < 5a_n + 10$.

3. $3^7 = 2187$ and $7! = 5040$. So $3^n < n!$ is true for $n = 7$.

Now suppose that $3^n < n!$ for some $n \in \mathbb{N}$ with $n \geq 7$. Then $3^{n+1} = 3 \times 3^n < 3 \times n! < (n+1) \times n! = (n+1)!$ So $3^n < n! \Rightarrow 3^{n+1} < (n+1)!$ for all $n \in \mathbb{N}$ with $n \geq 7$

So by induction $3^n < n!$ for all $n \in \mathbb{N}$ with $n \geq 7$

4.

a) $104 = 8 \times 13 = 2^3 \times 13$. So the *positive* divisors (this is what I meant) are 1, 2, 4, 8, 13, $26 = 2 \times 13$, $52 = 4 \times 13$ and $104 = 8 \times 13$.

b) $462 = 2 \times 231 = 2 \times 3 \times 77 = 2 \times 3 \times 7 \times 11$. So the positive divisors are 1, 2, 3, 7, 11 6, 14, 22, 21, 33, 42, 66, 77, 154, 231, 462.

c) $3432 = 8 \times 429 = 8 \times 3 \times 143 = 8 \times 3 \times 11 \times 13$. So the positive divisors are 2^n, $2^n \cdot 3$, $2^n \cdot 11$, $2^n \cdot 13$, $2^n \cdot 33$, $2^n \cdot 39$, $2^n \cdot 143$ and $2^n \cdot 429$, all for $0 \leq n \leq 3$, that is, writing them in increasing order.

$$1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 11,\ 12,\ 13,\ 22,\ 24,\ 26,\ 33,\ 39,\ 44,\ 52,\ ,\ 66,\ 78,\ 88,\ 104,\ 132,\ 143,\ 156,\ 264,$$

$$286,\ 312,\ 429,\ 572,\ 858,\ 1144,\ 1716,\ 3432.$$

The number of positive divisors is computed from the prime factorisation, thus, $(3+1) \times (1+1)$ in part a) and $(1+1) \times (1+1) \times (1+1) \times (1+1)$ in part b) and $(3+1) \times (1+1) \times (1+1) \times (1+1)$ in part c). I did want all the divisors written down, and I think all the answers that I saw did recognise this.

5. We have

$$\prod_{i=2}^{2} \left(1 - \frac{1}{i^2}\right) = 1 - \frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2 \times 2}$$

So the formula is true for $n = 2$. Now assume inductively that for some integer $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}.$$

Then

$$\prod_{i=2}^{n+1} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n} \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}(n+1)^2 - 1(n+1)^2$$

$$= \frac{1}{2n} \frac{n^2 + 2n}{n + 1} = \frac{n + 2}{2(n + 1)}$$

So if the formula holds for n it holds for $n + 1$ and hence by induction it holds for all n.

Unfamiliarity with product notation was, not surprisingly, a source of some difficulty with this question. Product notation is very similar to sum notation. So

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \cdot \times \left(1 - \frac{1}{n^2}\right)$$

and

$$\prod_{i=2}^{n+1} \left(1 - \frac{1}{i^2}\right) = \left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \cdot \times \left(1 - \frac{1}{n^2}\right) \times \left(1 - \frac{1}{(n + 1)^2}\right)$$

Solutions to Practice Problems

6. When $n = 2$

$$2^n + n^2 = 4 + 4 = 8 < 9 = 3^2$$

So $2^n + n^2 < 3^n$ is true for $n = 2$. Now assume it is true for some $n \in \mathbb{N}$ with $n \geq 2$. Then using $(n+1)^2 \leq \frac{9}{4}n^2$,

$$2^{n+1} + (n+1)^2 \leq 2^n + 2^n + \frac{9}{4}n^2 = 2 \times (2^n + n^2) + \frac{1}{4}n^2 < 3(2^n + n^2) < 3 \cdot 3^n = 3^{n+1}.$$

So $2^n + n^2 < 3^n \Rightarrow 2^{n+1} + (n+1)^2 < 3^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 2$. So by induction, $2^n + n^2 < 3^n$ for all $n \in \mathbb{N}$ with $n \geq 2$.

7. When $n = 0$ we have $a_0 = 2 = 3^0 + 1$. So $a_n = 3^n + 1$ is true for $n = 0$. Now assume that $a_n = 3^n + 1$ for some $n \in \mathbb{N}$. Then $a_{n+1} = 3a_n - 2 = 3(3^n + 1) - 2 = 3^{n+1} + 1$. So by induction the formula $a_n = 3^n + 1$ holds for all $n \in \mathbb{N}$.

8. We have $\frac{1}{2} < a_0 = 1$. So $\frac{1}{2} \leq a_n \leq 1$ when $n = 0$. Now assume this holds for some $n \in \mathbb{N}$. We want to deduce it for $n + 1$. If $a_n \geq \frac{1}{2}$ then $3a_n + 1 \geq \frac{5}{2} > 0$. and $3a_n + 1 > a_n + 1$. So it is certainly true that

$$\frac{a_n + 1}{3a_n + 1} < 1$$

Since $3a_n + 1 > 0$ we have

$$\frac{1}{2} \leq \frac{a_n + 1}{3a_n + 1} \Leftrightarrow 3a_n + 1 \leq 2a_n + 2 \Leftrightarrow a_n \leq 1.$$

So

$$a_n \leq 1 \Rightarrow \frac{1}{2} < a_{n+1}$$

and

$$\frac{1}{2} \leq a_n \leq 1 \Rightarrow \frac{1}{2} \leq a_{n+1} \leq 1$$

So by induction $\frac{1}{2} \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

9. From the definition of multiplication, and $m \cdot 1 = m$ we have

$$m \cdot (n + 1) = m \cdot n + m = m \cdot n + m \cdot 1$$

So $m \cdot (n + p) = m \cdot n + m \cdot p$ is true for $p = 1$.

Now assume inductively that it is true for p. Then

$$m \cdot (n + (p + 1)) = m \cdot ((n + p) + 1) = m \cdot (n + p) + m \cdot 1 = (m \cdot n + m \cdot p) + m.$$

The first equality uses associativity of addition and the second uses the inductive definition of multiplication and the third uses the inductive hypothesis and $m \cdot 1 = m$. But then

$$(m \cdot n + m \cdot p) + m = m \cdot n + (m \cdot p + m) = m \cdot n + m \cdot (p + 1)$$

where the first equality uses associativity of multiplication and the second uses the inductive definition of multiplication. This completes the proof that

$$(m \cdot (n + p) = m \cdot n + m \cdot p) \quad \Rightarrow \quad (m \cdot (n + (p + 1)) = m \cdot n + m \cdot (p + 1))$$

So by induction

$$m \cdot (n + p) = m \cdot n + m \cdot p$$

for all m, n, $p \in \mathbb{Z}_+$.