MATH105 Feedback and Solutions 2

1.

a) $-1 \notin \mathbb{N}$ because $\mathbb{N}$ is the set of natural numbers, that is, non-negative integers

b) Yes

c) No

d) Yes because $(-4,-1] = \{x \in \mathbb{R} : -4 < x \leq -1\}$

e) No because $(-1,\infty) = \{x \in \mathbb{R} : x > -1\}$.

2.

a) The statement says that $n \geq 1$ for all $n \in \mathbb{N}$. This is false because $0 \in \mathbb{N}$ and $0 < 1$.

b) The statement says that there exists $n \in \mathbb{N}$ with $n \leq 1$. This is true because 0 and $0 \leq 1$ and $0 \leq 1$. Alternatively, we can use the fact that $1 \in \mathbb{N}$ and $1 \leq 1$ to show that the statement is true.

c) The statement says that *for all* n and $m \in \mathbb{N}$ with $n < m$, there exists $p \in N$ with $m < p < n$. This is false because although p exists for *some* choices of n and $m \in \mathbb{N}$ with $n < m$ – such as $n = 0$ and $m = 2$ – it is not true for *all* choices. If we take $n = 0$ and $m = 1$ then $n \in \mathbb{N}$ and $m \in \mathbb{N}$, but for any $p \in \mathbb{N}$, $p = 0 \vee p = 1 \vee p > 1$. In each of these cases it is not true that $0 < p < 1$. So if $m = 1$ and $n = 1$, there is no $p \in N$ with $m < p < n$.

d) This time, the statement is true. For any x and $y \in \mathbb{Q}$ with $x < y$, if we take $z = \dfrac{x+y}{2}$ then $z \in \mathbb{Q}$ and $x < z < y$, because $z - x = \dfrac{y-x}{2} > 0$ and $y - z = \dfrac{y-x}{2} > 0$.

e) This statement is actually equivalent to the statement in a). It says that if $n \in \mathbb{N}$ then $n \geq 1$. As for the statement in a), this is false because $0 \in \mathbb{N}$ and $0 < 1$.

3.

a) The negation is " $\exists n \in \mathbb{N}$ such that $n < 1$". (This is true because 2a) is false.)

b) The negation is "$\forall n \in \mathbb{N}$, $n > 1$. (This is false because 2b) is true)

e) The negation is " $\exists n \in \mathbb{N}$ such that $n < 1$". (This is true because 2e) is false, and is equivalent to the statement in 3a) because 2a) and 2e) are equivalent.)

As a rough rule, the negation of a statement "$\exists n, A(n)$" is "$\forall n, \neg A(n)$", and the negation of "$\forall n, A(n)$" is "$\exists n, \neg A(n)$. Also the negation of "$A(n) \Rightarrow B(n)$" is $\exists n, A(n) \wedge \neg B(n)$. Note that the comma usually means "such that" when it follows "There exists.. " but not when if follows "For all .." For example, "$\exists n \in \mathbb{N}$ such that $n < 1$" is natural but "$\forall n \in \mathbb{N}$, such that $n > 1$" is not only unnatural but does not make sense. That is why I have crossed out "such that" in quite a few answers to 2b). Always think what the statement says in ordinary English and check that what you write symbolically is in natural language.

4.

a)

$$x^2 \geq 16 \Leftrightarrow x^2 - 16 = (x-4)(x+4) \geq 0$$

$$\Leftrightarrow (x-4 \geq 0 \wedge x+4 \geq 0) \vee (x-4 \leq 0 \wedge x+4 \leq 0) \Leftrightarrow x \geq 4 \vee x \leq -4 \Leftrightarrow x \in (-\infty - 4] \cup [4, \infty)$$

b)

$$x^2 + 5x - 6 \geq 0 \Leftrightarrow (x+6)(x-1) \geq 0$$

$$\Leftrightarrow (x+6 \geq 0 \wedge x-1 \geq 0) \vee (x+6 \leq 0 \wedge x-1 \leq 0)$$

$$\Leftrightarrow (x \geq -6 \wedge x \geq 1) \vee (x \leq -6 \wedge x \leq 1)$$

$$\Leftrightarrow x \geq 1 \vee x \leq -6 \Leftrightarrow x \in [1, \infty) \cup (-\infty, -6]$$

An alternative acceptable mehtod is completing the square. This works as follows.

$$x^2 + 5x - 6 = (x + \frac{5}{2})^2 - \frac{49}{4} \geq 0 \Leftrightarrow \left| x + \frac{5}{2} \right| \geq \frac{7}{2}$$

$$\Leftrightarrow x + \frac{5}{2} \leq -\frac{7}{2} \vee \frac{7}{2} \leq x + \frac{5}{2} \Leftrightarrow x \leq -6 \vee 1 \leq x.$$

c) Square the original inequality:

$$\left| \frac{3x+2}{2x+3} \right| < 1 \Leftrightarrow \frac{(3x+2)^2}{(2x+3)^2} < 1 \Leftrightarrow (3x+2)^2 < (2x+3)^2$$

$$\Leftrightarrow 9x^2 + 12x + 4 < 4x^2 + 12x + 9 \Leftrightarrow 5x^2 < 5 \Leftrightarrow -1 < x < 1.$$

Full marks for this question required some reasoning, although I was generous on part a). Do be careful to write in proper sentences, using implication signs to connect different lines in your solution. A correct version, which clearly has something in common with this, is given in the solutions above. The statement "$x \in [1, \infty) \cup (-\infty, -6]$" means "$x$ is in the union of intervals $[1, \infty)$ and $(-\infty, -6]$"

The need for reasoning in part b) was clear because some people who did not write anything much down often got the answer wrong, saying, for example, that the answer was $-6 \leq x \leq 1$. Alternatives to the answer I have given are: to complete the square or draw a graph: which I accepted if there was adequate commentary and the graph had enough detail. The best type of answer using a graph would say that the graph is a parabola with a minimum at $-5/2$ and crossing thex-axis at $x = -6$ and $x = 1$.

In part c), the squaring method comes out very easily for this particular example, because the equation reduces to $x^2 < 1$. (But look at the corresponding tutorial problem, where this method takes a bit more work, and it should be recognised, when factorising a quadratic, that factors can be negative as well as positive.) Some people tried to do it without squaring but none that I saw was completely correct. Note that $|\frac{3x+2}{2x+3}| \leq 1 \Leftrightarrow -1 < \frac{3x+2}{2x+3} < 1$ and those two inequalities both *have to hold and so have to be treated separately. Also, one has to consider separately the cases $2x+3 > 0$ and $2x+3 < 0$, because multiplying through by negative numbers reverses inequalities. This has to be done, even though there are no solutions with $2x+3 < 0$. It is impossible to tell that in advance.*

5. $2^2 = 4 < 3 = 3^2$. So $n^2 < 3^n$ is true for $n = 2$.

Now assume that $n^2 < 3^n$ for some $n \in \mathbb{N}$ with $n \geq 2$. Then

$$(n+1)^2 = n^2(1+\frac{1}{n})^2 \leq n^2(1+\frac{1}{2})^2 < 2\frac{1}{4} \cdot 3^n < 3^{n+1}$$

for all $n \geq 2$. So for $n \geq 2$ we have

$$n^2 < 3^n \Rightarrow (n+1)^2 < 3^{n+1}.$$

So by induction $n^2 < 3^n$ for all $n \geq N$ with $n \geq 2$.

This is also true for $n = 0$ and $n = 1$ because $0^2 = 0 < 1 = 3^0$ and $1^2 = 1 < 3^1 = 3$.

The base case in this example is $n = 2$. That is taken as the base case because the inductive step works more easily when $n \geq 2$. The inequality does hold when $n = 0$ and $n = 1$ also, which is why you were asked to check these cases separately.

Solutions to Practice Problems

6.

a) No (because $1.5 = \frac{3}{2}$ is not an integer)

b) Yes (because 1.5 is a real number).

c) Yes (because $1 < 1.5 < 2$).

7.

a) False because $0 \in \mathbb{N}$ and it is not true that $0 > 0$.

b) True because $0 \in \mathbb{N}$ and $0 \leq 0$.

c) True because it $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$ and $n < n+1$. So given $n \in \mathbb{N}$, we can take $m = n+1$, and then $n < m$.

d) True, because given $x \in \mathbb{Q}$ with $0 < x$, if we define $y = x/2$, then $y \in \mathbb{Q}$ and $0 < y < x$.

8. Statement b) is the negation of a) and a) is the negation of b). The negation of a true statement is false and the negation of a false statement is true. This is consistent with the answers to 7, since a) is a false statement and b) is a true statement. This is not to say that any false statement is the negation of any true statement, or that any true statement is the negation of any false statement!

9.

a) Since $2.25 = \frac{9}{4} = \frac{3^2}{2^2}$, we have

$$x^2 \leq 2.25 \Leftrightarrow x^2 \leq \left(\frac{3}{2}\right)^2 \Leftrightarrow -\frac{3}{2} \leq x \leq \frac{3}{2}.$$

b) $x^2 - 3x - 4 = (x+1)(x-4)$. So

$$x^2 - 3x - 4 \geq 0 \Leftrightarrow (x+1)(x-4) \geq 0 \Leftrightarrow ((x+1 \geq 0 \wedge x-4 \geq 0) \vee (x+1 \leq 0 \wedge x-4 \leq 0))$$

$$\Leftrightarrow ((x \geq -1 \wedge x \geq 4) \vee (x \leq -1 \wedge x \leq 4)) \Leftrightarrow x \geq 4 \vee x \leq -1.$$

c)

$$\left|\frac{3x-1}{x+2}\right| < 1 \Leftrightarrow -1 < \frac{3x-1}{x+2} < 1$$

$$\Leftrightarrow ((x+2>0 \wedge -x-2 < 3x-1 < x+2) \vee (x+2<0 \wedge -(x+2) > 3x-1 > x+2))$$

$$\Leftrightarrow ((4x > -1 \wedge x < \frac{3}{2}) \vee (x < -1/4 \wedge x > \frac{3}{2}))$$

$$\Leftrightarrow -1/4 < x < \frac{3}{2}$$

because there are no x satisfying $x < -1/4 \wedge x > \frac{3}{2}$.

An **alternative method** is to square the original inequality:

$$\left|\frac{3x-1}{x+2}\right| < 1 \Leftrightarrow \frac{(3x-1)^2}{(x+2)^2} < 1 \Leftrightarrow (3x-1)^2 < (x+2)^2$$

$$\Leftrightarrow 8x^2 - 10x - 3 < 0 \Leftrightarrow (4x+1)(2x-3) \Leftrightarrow (4x+1>0 \wedge 2x-3>0) \vee (4x+1<0 \wedge 3x-2<0)$$

$$\Leftrightarrow x > \frac{3}{2} \vee x < -\frac{1}{4}$$

10. When $n = 2$,

$$2n+1 = 5 < 8 = 2n^2$$

So $2n+1 < 2n^2$ is true for $n = 2$. Now assume it is true for some $n \in \mathbb{N}$ with $n \geq 2$. Then

$$2(n+1)+1 = (2n+1)+2 < 2n^2+2 < 2n^2+4n+2 = 2(n+1)^2$$

So $2n+1 < 2n^2 \Rightarrow 2(n+1)+1 < 2(n+1)^2$ for all $n \in \mathbb{N}$ with $n \geq 2$. So $2n+1 < 2n^2$ for all $n \in \mathbb{N}$ with $n \geq 2$.