Solutions 11

1.

a) Since $n^2+1<(n+1)^2+1$ for all $n\geq 1$, we have $\dfrac{2}{(n+1)^2+1}<\dfrac{2}{n^2+1}$ for all $n\geq 1$ and $x_{n+1}<x_n$ for all $n>1$. So x_n is decreasing.

b) Since $\dfrac{2}{(n+1)^2+1}<\dfrac{2}{n^2+1}$ for all $n\geq 1$ we have $1-\dfrac{2}{n^2+1}<1-\dfrac{2}{(n+1)^2+1}$ for all $n\geq 1$. So this sequence x_n is increasing.

c) We have $x_n=n^2-3n+1=(n-1)(n-2)$. So $x_1=x_2=0$ and for $n\geq 2$, the numbers $n-1$ and $n-2$ are increasing with n and positive. So the product $(n-1)(n-2)$ is postive and increasing with n, and strictly positive for $n\geq 3$. Sop $x_n\leq x_{n+1}$ for all $n\geq 1$. in fact $x_n<x_{n+1}$ for all $n\geq 2$, and x_n is an increasing sequence.

d) $3^n<3^{n+1}$ for all $n\geq 1$ and $1-3^{n+1}<1-3^n<0$ for all $n\geq 1$. So $\dfrac{1}{1-3^n}<\dfrac{1}{1-3^{n+1}}$ for all $n\geq 1$ and this sequence x_n is increasing

e) $x_{n+1}-x_n=\dfrac{1}{(n+1)!}>0$ so $x_n<x_{n+1}$ and x_n is increasing. (Looking at the first few terms, $x_1=1$, $x_2=\frac{3}{2}$, $x_3=\frac{5}{3}$, $x_4=\frac{41}{24}$.)

f) $x_{n+1}-x_n=\dfrac{(-1)^{n+2}}{(n+1)^2}$ which is >0 if n is even and <0 if n is odd. So this sequence x_n is neither increasing nor decreasing. Alternatively, we can just look at the first few terms. We have $x_1=1$, $x_2=1-\dfrac{1}{4}=\dfrac{3}{4}$ and $x_3=1-\dfrac{1}{4}+\dfrac{1}{9}=\dfrac{31}{36}>\dfrac{3}{4}$. So $x_2<x_3<x_1$ and the sequence is neither increasing nor decreasing.

g) $x_1=1$, $x_2=\dfrac{1}{1+1}=\dfrac{1}{2}$, $x_3=\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}$. So $x_2<x_3<x_1$ and the sequence is neither increasing nor decreasing.

h)

$$x_1=1,\quad x_2=\frac{1}{1+1}=\frac{1}{2},\quad x_3=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3},\quad x_4=\frac{\frac{1}{3}}{1+\frac{1}{3}}=\frac{1}{4}.$$

It seems reasonable to conjecture that $x_n=\frac{1}{n}$ for all $n\geq 1$. This is certainly true for $n=1$. Suppose inductively that $x_n=\dfrac{1}{n}$. Then

$$x_{n+1}=\frac{\frac{1}{n}}{1+\frac{1}{n}}=\frac{1}{n}\cdot\frac{n}{n+1}=\frac{1}{n+1}.$$

So if true for n it is true for $n+1$ and hence $x_n=\dfrac{1}{n}$ for all $n\geq 1$. This is a decreasing sequence.

2.

a) This is a decreasing sequence of strictly positive numbers with $x_1 = \dfrac{1}{2}$. So the sequence is bounded above by $\dfrac{1}{2}$ and below by 0.

b) This is an increasing sequence of numbers, all less than 1, with $x_1 =$. So the sequence is bounded above by 1 and below by 0.

c) Since both $n-1$ and $n-2$ get arbitrarily large and positive as $n \to \infty$, the sequence $x_n(n-1)(n-2)$ is not bounded above. However it is an increasing sequence and hence it is bounded below by $x_1 = 0$.

d) This is an increasing sequence of negative numbers, bounded below by $x_1 = -\dfrac{1}{2}$ and above by 0.

e) Since $k!$ is the product of 1 and $k-1$ positive numbers which are all ≥ 2, we have $k! \geq 2^{k-1}$. Hence

$$x_n = \sum \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}} = \sum_{k=0}^{n-1} \frac{1}{2^k} = 2 - \frac{1}{2^{n-1}} < 2$$

Since x_n is clearly positive for all n, the sequence is bounded above by 2 and below by 0.

f) We see that

$$x_{n+1} - x_n = \frac{(-1)^{n+2}}{(n+1)^2} \tag{1}$$

which is < 0 if n is odd and > 0 if n is even. So $x_{n+1} < x_n$ if n is odd and > 0 if n is even. Also from (1), it follows that $0 < x_{n+2} - x_{n+1} < x_n - x_{n+1}$ whenever n is odd, that is, $x_{n+1} < x_{n+2} < x_n$ whenever n is odd. Similarly $x_n < x_{n+2} < x_{n+1}$ whenever n is even. It follows that the sequence x_n is bounded below by $x_2 = \dfrac{3}{4}$ and above by $1 = x_1$.

g) By induction $x_n > 0$ for all n because $x_1 = 1 > 0$ and if $x_n > 0$ then $x_{n+1} = \dfrac{1}{1+x_n} > 0$. So $0 < 1 + x_n$ for all $n \geq 1$ and $x_{n+1} < 1$ for all $n \geq 1$. Also, $x_1 = 1$. So x_n is bounded above by 1 and below by 0

h) Using $x_n = \frac{1}{n}$ we see that x_n is bounded above by 1 and below by 0. Alternatively, if one has not noticed this, one can argue as follows:

By induction $x_n > 0$ for all n because $x_1 = 1 > 0$ and if $x_n > 0$ then $x_{n+1} = \dfrac{x_n}{1+x_n} > 0$. So $0 < x_n < 1 + x_n$ for all $n \geq 1$ and $x_{n+1} < 1$ for all $n \geq 1$. Also, $x_1 = 1$. So x_n is bounded above by 1 and below by 0

3. The sequences of 1a), 1b), 1d), 1e) and 1h) are all either increasing or decreasing, and bounded (both above and below). By the Completeness Axiom, any bounded sequence of real numbers which is either increasing, or decreasinf, has a limit in $\mathbb{R}$.

The sequence in c) does not have a limit because it is not bounded and any sequence which has a limit must be bounded.

4.

a) It is shown above in the solution to question 2 that the sequence in 1f) satisfies $x_{2n} < x_{2n+1} < x_{2n-1}$ for all $n \geq 1$.

We also saw in the solution to question 2 that for the sequence 1g), $x_2 < x_3 < x_1$. Now we prove by induction on n that $x_{2n} < x_{2n+1} < x_{2n-1}$ for all $n \geq 1$. We know it is true for $n = 1$. So now we assume it is true for n and prove if for $n+1$. It suffices to show that

$$x_n < x_m \Rightarrow x_{m+1} < x_{n+1}$$

because then

$$x_{2n} < x_{2n+1} < x_{2n-1} \Rightarrow x_{2n} < x_{2n+2} < x_{2n+1} \Rightarrow x_{2n+2} < x_{2n+3} < x_{2n+1}$$

which is what we need to prove. We know from the solution to 2 that $x_n > 0$ for all n

$$0 < x_n < x_m \Rightarrow 1 < 1 + x_n < 1 + x_m \Rightarrow x_{m+1} = \frac{1}{1+x_m} < \frac{1}{1+x_n} = x_{n+1} < 1$$

which is what we needed to prove.

b) Since x_{2n-1} is a decreasing sequence bounded below by x_2, it has a real limit ℓ_1 and since x_{2n} is an increasing sequence bounded above by x_1 it has a real limit limit ℓ_2.

c) In case 1f)

$$x_{2n-1} - x_{2n} = \frac{1}{(2n)^2} = \frac{1}{4n^2} \tag{2}$$

So

$$\lim_{n\to\infty} x_{2n-1} - x_{2n} = \lim_{n\to\infty} \frac{1}{4n^2} = 0$$

Given $\varepsilon > 0$ there exists N such that for all $n \geq N$

$$|x_{2n-1} - \ell_1| \leq \varepsilon, \quad |x_{2n} - \ell_2| < \varepsilon, \quad |x_{2n-1} - x_{2n}| = \frac{1}{4n^2} < \varepsilon.$$

So

$$|\ell_1 - \ell_2| \leq |\ell_1 - x_{2n-1}| + |x_{2n-1} - x_{2n}| + |x_{2n} - \ell_2| < 3\varepsilon.$$

Since ε can be taken as small as we like it follows that $\ell_1 = \ell_2$ and hence for all $n \geq N$, $|x_n - \ell_1| < \varepsilon$. So $\lim_{n\to\infty} x_n = \ell_1$.

d) It is reasonable to expect that a limit ℓ would satisfy

$$x = \frac{1}{1+x}$$

which implies $x^2 - x - 1 = 0$ and $x = (1 \pm \sqrt{5})/2$. Since we know that $x_n > 0$ for all n, we guess that the limit is $(-1 + \sqrt{5})/2$. This is true.

5.

a) $x \mapsto \ln x$ maps $(0,\infty)$ onto $\mathbb{R}$ and $\mathbb{R}$ is uncountable, so $(0,\infty)$ is uncountable.

b) $x \mapsto \frac{1}{x} - 1$ maps $(0,1)$ ontoto $(0,\infty)$. If we define

$$f(x) = \begin{cases} \frac{1}{x} - 1 \text{ if } x \in (0,1), \\ 1 \text{ if } x = 0 \text{ or } 1 \end{cases}$$

then f maps $[0,1]$ onto $(0,\infty)$ which is uncountable by a) and hence $[0,1]$ is uncountable.

c) Since $\mathbb{Q}$ itself is countable, any subset is also countable and in particular $\mathbb{Q} \cap [0,1]$ is countable.

d) Since the set of all integers is countable the subset of even integers is countable. It is also easy to construct a bijection f from $\mathbb{Z}_+$ onto the set of even integers by

$$f(n) = \begin{cases} n \text{ if } n \text{ is even} \\ 1 - n \text{ if } n \text{ is odd} \end{cases}$$

6. The map

$$((a_1, \cdots a_{n-1}), a_n) \mapsto (a_1, \cdots a_n)$$

is a bijection between $A^{n-1} \times A$ and A^n. So $A^{n-1} \times A$ is countable $\Leftrightarrow$ A^n is countable.

$A^1 = A$ is countable. Assume inductively that A^n is countable. Then putting $B = A^n$, $B \times A = A^n \times A$ is countable and hence A^{n+1}, which is bijective to $A^n \times A$, is countable. Hence A^n is countable for all n.

7. Suppose that $X = \{(a_n) : a_n \in A \forall n \in \mathbb{Z}_+\}$ is countable. then there is a bijection $f : \mathbb{Z}_+ \to X$. Write $f(n) = (a_{m,n})$. Choose a sequence (a_m) so that $a_m \neq a_{m,m}$ for each m. This is possible because A has at least two elements and therefore we can always choose $a_m \in A \setminus \{a_{m,m}\}$. Since $a_n \neq a_{n,n}$ for each $n \in \mathbb{Z}_+$, $(a_m) \neq f(m)$ for each $m \in \mathbb{Z}_+$. But $(a_n) \in X$ and hence f is not surjective. This is a contradiction, because f is a bijection.

8.

1. If $x = 3 - \sqrt{2}$, then $x^2 = (3 - \sqrt{2})^2 = 11 - 6\sqrt{2}$ and so $x^2 - 6x + 7 = 0$
2. If $x = \sqrt{3} - 1$ then $x^2 = 4 - 2\sqrt{3}$ and so $x^2 + 2x - 2 = 0$
3. If $x = \sqrt{3} - \sqrt{2}$ then $x^2 = 5 - 2\sqrt{6}$ and so $x^4 = 49 - 20\sqrt{6}$ and $x^4 - 10x^2 + 1 = 0$.