MATH105 Feedback and Solutions 10

1.

a) Because of the way the continued fraction expansion repeats, we need a number x satisfying

$$x = \frac{1}{3+x}$$

that is

$$x^2 + 3x - 1 = 0.$$

This implies that

$$x = -\frac{3}{2} \pm \frac{\sqrt{13}}{2}$$

Since all continued fractions with positive integers represent positive numbers, we must have $x = (-3+\sqrt{13})/2$.

b) This time we must have

$$x = \frac{1}{3+\frac{1}{1+x}} = \frac{x+1}{3x+4}.$$

So

$$3x^2 + 3x - 1 = 0$$

and

$$x = \frac{-3 \pm \sqrt{21}}{6}$$

and again we need to take the positive root. So $x = (-3+\sqrt{21})/2$

2.

a) One could use calculus, but it is not necessary because if $x < y$ then $x^3 < y^3$ and hence $x^3 + x + 3 < y^3 + y + 3$. If using calculus, then $f'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, and hence f is strictly increasing.

b) There are no integer solutions to $f(x) = 0$ because $f(-2) = -7$ and $f(-1) = 1$. So $f(n) < 0$ for all $n \in Z$ with $n \leq -2$ and $f(n) > 0$ for all $n \in \mathbb{Z}$ with $n \geq -1$. Suppose

$$\frac{p^3}{q^3} + \frac{p}{q} + 3 = 0$$

for $p \in Z$ and $q \in \mathbb{Z}_+$. We can assume the g.c.d of p and q is one and then $q \geq 2$ because there are no integer solutions to $f(x) = 0$. Then multiplying by q^3 we have

$$p^3 + pq^2 + 3q^3 = 0$$

This can be rewritten as

$$p^3 = -q^2(p + 3q)$$

Let k be any prime factor of q. There is at least one, because $q \geq 2$.Then $k|p^3$. Hence by unique factorisation, $k|p$ and k is a factor of both p and q, giving a contradiction.

It is necessary to take a prime dividing q. If q itself is not prime then one cannot deduce, from $q \mid p^3$, that $q \mid p$. For example, $4 \mid 6^3 = 216$, but $4 \not| 6$.

c) The set $A = \{x \in \mathbb{Q} : x^3 + x + 3 < 0\}$ is a Dedekind cut because $-2 \in A$, $0 \notin A$ and $x \in A \wedge y < x \Rightarrow f(y) < f(x) < 0 \Rightarrow y \in A$ (because f is strictly increasing) and A has no maximal element - which can be proved using continuity of f.

You were not required to prove that A is a Dedekind cut in this exercise. But it should be made clear that A is a set of rational *numbers. Remember that $\mathbb{Q}$ is the set of rational numbers.*

3.

a) For $f(x) = x^3 - 12x + 2$,

$$f(-4) = -14 < 0, \quad f(-3) = 11 > 0, \quad f(0) = 2 > 0, \quad f(1) = -9 < 0,$$

$$f(3) = -7 < 0, \quad f(4) = 14 > 0.$$

Applying the intermediate value theorem to f on each of the intervals $[-4,-3]$, $[0,1]$ and $[3,4]$, we see that f has a zero in each of the intervals $(-4,-3)$, $(0,1)$ and $(3,4)$. Also $f'(x) = 3x^2 - 12 = 3(x^2-4) = 0 \leftrightarrow x = \pm 2$. Also $f'(x) > 0$ if $x \in (-\infty,-2) \cup [(2,\infty)$ and $f'(x) < 0$ on $(-2,2)$. So f is strictly increasing on each of the intervals $(-\infty,-2]$ and on $[2,\infty)$, and strictly decreasing on $[-2,2]$. In particular f is strictly increasing on each of the intervals $[-4,-3]$ and $[3,4]$ and strictly decreasing on $[0,1]$. So because of the values of f that have been computed, f must have a zero in each of the intervals $(-4,-3)$, $(3,4)$ and $(0,1)$.

The change of sign on each of the intervals, and the Intermediate Value Theorem, show that there is at least one *zero of f in each of the intervals $(-4,-3)$, $(0,1)$, $(3,4)$. The calculus is used to show that there are* no more than *three zeros, by showing that there is* at most one *zero of f in each of the intervals $(-4,-3)$, $(0,1)$, $(3,4)$. It is acceptable to say that any cubic polynomial has at most three zeros.*

b) The Dedekind cuts can be expressed as

$$A_1 = \{x \in \mathbb{Q} : f(x) < 0 \wedge x < -3\}, \quad A_2 = \{x \in \mathbb{Q} : x < -3\} \cup \{x \in \mathbb{Q} : f(x) > 0 \wedge x < 1\},$$

$$A_3 = \{x \in \mathbb{Q} : x < 3 \vee f(x) < 0\}$$

In each case, $x \in A_j \wedge y < x \Rightarrow y \in A_j$, $-4 \in A_j$, $5 \notin A_j$ and A_j has no maximal element. Full proof of A_j not having a maximal element is not required.

There are many forms for the correct solution.

4.

$p_{-1}q_0 - p_0q_{-1} = 1 - 0 = 1 = (-1)^0$. So $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ is true for $n = 0$.
Now assume that $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$. Then

$$p_nq_{n+1} - p_{n+1}q_n = p_n(q_{n-1} + a_nq_n) - (p_{n-1} + a_np_n)q_n$$

$$= p_nq_{n-1} + a_np_nq_n - p_{n-1}q_n - a_np_nq_n = p_nq_{n-1} - p_{n-1}q_n = -(-1)^n = (-1)^{n+1}.$$

So by induction $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ for all $n \geq 0$.

This proof uses the inductive definition of p_{n+1} and q_{n+1}, that is $p_{n+1} = p_{n-1} + a_{n+1}p_n$, and similarly for q_{n+1}. This is what the hint suggested. You should deduce *that $p_nq_{n+1} - p_{n+1}q_n = (-1)^{n+1}$* from the assumption *that $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$.*

Solutions to Practice Problems

5.

a) Because of the way the continued fraction expansion repeats, we need a number x satisfying

$$x = \frac{1}{4+x}$$

that is

$$x^2 + 4x - 1 = 0.$$

This implies that

$$x = -2 \pm \sqrt{5}$$

Since all continued fractions with positive integers represent positive numbers, we must have $x = -2 + \sqrt{5}$.

b) This time we must have

$$x = \frac{1}{4 + \frac{1}{1+x}} = \frac{x+1}{4x+5}.$$

So

$$4x^2 + 4x - 1 = 0$$

and

$$x = \frac{-2 \pm \sqrt{8}}{4} = \frac{-1 \pm \sqrt{2}}{2}$$

and again we need to take the positive root. So $x = (-1 + \sqrt{2})/2$

6.

a) One could use calculus, but it is not necessarily because if $x < y$ then $x^3 < y^3$ and hence $x^3 + 2x + 5 < y^3 + 2y + 5$. If using calculus, then $f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$, and hence f is strictly increasing.

b) There are no integer solutions to $f(x) = 0$ because $f(-2) = -7$ and $f(-1) = 2$. So $f(n) < 0$ for all $n \in Z$ with $n \leq -2$ and $f(n) > 0$ for all $n \in \mathbb{Z}$ with $n \geq -1$. Suppose

$$\frac{p^3}{q^3} + 2\frac{p}{q} + 5 = 0$$

for $p \in Z$ and $q \in \mathbb{Z}_+$. We can assume the g.c.d of p and q is one and then $q \geq 2$ because there are no integer solutions to $f(x) = 0$Then multiplying by q^3 we have

$$p^3 + 2pq^2 + 5q^3 = 0$$

This can be rewritten as

$$p^3 = -q^2(2p + 5q)$$

Let k be any prime factor of q. There is at least one because $q \geq 2$.Then $k|p^3$. Hence by unique factorisation, $k|p$ and k is a factor of both p and q, giving a contradiction.

c) The set $A = \{x \in \mathbb{Q} : x^3 + 2x + 5 < 0\}$ is a Dedekind cut because it has no maximal element, $0 \notin A$ and $x \in A \wedge y < x \Rightarrow f(y) < f(x) < 0 \Rightarrow y \in A$.

7.

a) For $f(x) = x^3 - 12x + 1$,

$$f(-4) = -15 < 0, \quad f(-3) = 10 > 0, \quad f(0) = 1 > 0, \quad f(1) = -10 < 0,$$

$$f(3) = -8 < 0, \quad f(4) = 15 > 0.$$

Applying the intermediate value theorem to f on each of the intervals $[-4, -3]$, $[0, 1]$ and $[3, 4]$, we see that f has a zero in each of the intervals $(-4, -3)$, $(0, 1)$ and $(3, 4)$. Also $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 0 \leftrightarrow x = \pm 2$. Also $f'(x) > 0$ if $x \in (-\infty, -2) \cup [(2, \infty)$ and $f'(x) < 0$ on $(-2, 2)$. So f is strictly increasing on each of the intervals $(-\infty, -2]$ and on $[2, \infty)$, and strictly decreasing on $[-2, 2]$. In particular f is strictly increasing on each of the intervals $[-4, -3]$ and $[3, 4]$ and strictly decreasing on $[0, 1]$. So because of the values of f that have been computed, f must have a zero in each of the intervals $(-4, -3)$, $(3, 4)$ and $(0, 1)$.

b) The Dedekind cuts can be expressed as

$$A_1 = \{x \in \mathbb{Q} : f(x) < 0 \wedge x < -3\}, \quad A_2 = \{x \in \mathbb{Q} : x < -3\} \cup \{x \in \mathbb{Q} : f(x) > 0 \wedge x < 1\},$$

$$A_3 = \{x \in \mathbb{Q} : x < 3 \vee f(x) < 0\}$$

In each case, $x \in A_j \wedge y < x \Rightarrow y \in A_j$ and $5 \notin A_j$ and A_j has no maximal element. Full proof of A_j not having a maximal element is not required.