

MATH105 Problem Sheet 8: Equivalence relations and rational numbers

Due Wednesday 26 November

1. Determine which of the following relations \sim are equivalence relations on the set X . For any which are equivalence relations, determine all the equivalence classes.

a) $X = \mathbb{Z}$, and \sim is defined by $m \sim n \Leftrightarrow 10|(m - n)$.

b) $X = \mathbb{Z}$, and $m \sim n \Leftrightarrow mn = 1$.

c) $X = \mathbb{Z}$ and $m \sim n \Leftrightarrow$ there is an integer $k \geq 2$ depending on m and n such that $k^2|(m - n)$.

2. Let Y be a nonempty finite set and let $X = 2^Y$, the set of subsets of Y . Let \sim be defined on X by $A \sim B \Leftrightarrow |A| = |B|$

a) Show that \sim is an equivalence relation on X .

b) In the special case $Y = \{1, 2, 3, 4\}$, write down all the equivalence classes of \sim .

c) In the general case $|Y| = n$ for $n \in \mathbb{Z}_+$, determine how many equivalence classes of \sim there are, and how many elements of X there are in each equivalence class. (*Caution:* the number of elements in different equivalence classes is not usually the same.)

3.

a) Find the least possible $n \in \mathbb{Z}_+$ such that $\{\frac{r}{n} : r \in \mathbb{Z}\}$ contains $\{\frac{k}{m} : k \in \mathbb{Z}, m \in \{4, 15, 35\}\}$.

b) For n as in a), find $a, b, c \in \mathbb{Z}$ where

$$\frac{1}{n} = \frac{a}{4} + \frac{b}{15} + \frac{c}{35}.$$

Hint. Write $n = 4 \times n_1 = 15 \times n_2 = 35 \times n_3$ and then find $a, b, c \in \mathbb{Z}$ such that $1 = an_1 + bn_2 + cn_3$. Since $n_1 = dm_1$ and $n_2 = dm_2$ where d is the g.c.d. of n_1 and n_2 and d and n_3 are coprime, this is equivalent to finding a_1, b_1, a_2 and $b_2 \in \mathbb{Z}$ such that

$$a_1m_1 + b_1m_2 = a_2d + b_2n_3 = 1.$$

Since m_1, m_2 and n_3 are not large, you can probably do this by inspection.

4. Assuming that $\sqrt{2}$ is not rational, show that:

a) $\frac{1}{3} + \frac{2}{5}\sqrt{2}$ is not rational;

b) $a + b\sqrt{2}$ is not rational, for any $a, b \in \mathbb{Q}$ with $b \neq 0$;

c) $2^{1/4}$ is not rational, where this is defined to be the real number $x > 0$ with $x^4 = 2$.

I will collect solutions at the lecture on Wednesday 26th November. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also be used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

5. Determine which of the following relations \sim are equivalence relations on the set X .

a) $X = \mathbb{R}$, and \sim is defined by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

b) $X = \mathbb{R}$, and $x \sim y \Leftrightarrow x - y \in \mathbb{N}$.

c) $X = \mathbb{Z}_+$ and $m \sim n \Leftrightarrow m/n \in \mathbb{Z}_+$

6. Let X denote the set of all vectors (m, n) with $m, n \in \mathbb{Z}$. Show that \sim is an equivalence relation where \sim is defined by

$$(m_1, n_1) \sim (m_2, n_2) \Leftrightarrow m_1 - m_2 \text{ even} \wedge n_1 - n_2 \text{ even} .$$

Write down a representative of each equivalence class. (There are only finitely many.)

7.

a) Find the least possible $n \in \mathbb{Z}_+$ such that $\{\frac{r}{n} : r \in \mathbb{Z}\}$ contains $\{\frac{k}{m} : k \in \mathbb{Z}, m \in \{11, 15, 35\}\}$.

b) For n as in a), find $a, b, c \in \mathbb{Z}$ where

$$\frac{1}{n} = \frac{a}{11} + \frac{b}{15} + \frac{c}{35}.$$

Hint. $n = \text{l.c.m.}(11, n_1)$ where $n_1 = \text{l.c.m.}(15, 35)$. So one possibility is to find b_1 and c_1 such that

$$\frac{b_1}{15} + \frac{c_1}{35} = \frac{1}{n_1}$$

and then find a and e such that

$$\frac{a}{11} + \frac{e}{n_1} = \frac{1}{n}.$$

Then we can take $b = b_1 e$ and $c = c_1 e$.

8. Assuming that $\sqrt{3}$ is not rational, show that:

a) $\frac{1}{7} + \frac{3}{4}\sqrt{3}$ is not rational;

b) $a + b\sqrt{3}$ is not rational, for any $a, b \in \mathbb{Q}$ with $b \neq 0$;

c) $3^{1/6}$ is not rational, where this is defined to be the real number $x > 0$ with $x^4 = 3$.

9. Let x_n be defined for $n \in \mathbb{N}$ by $x_0 = x_1 = 2$ and $x_{n+1} = 2x_n + 3x_{n-1}$ for $n \geq 1$. Prove by induction on n that $x_n = 3^n + (-1)^n$ for all $n \in \mathbb{N}$.

Hint. This is one of the ones where there are two base cases to verify, for $n = 0$ and $n = 1$, and then the inductive hypothesis is that $x_k = 3^k + (-1)^k$ for all $k \leq n$. This is because this is needed for at least $k = n - 1$ and $k = n$ in order to prove it for $k = n + 1$.