MATH105 Problem Sheet 6: Sets and Maps

Due Wednesday 12th November

1. Write the following sets as simply as possible in terms of intervals. If you are not sure of the answer, partial credit will be given for correct reasoning, even if the answer is wrong. Note that $A \setminus B$ means $\{x \in A : x \notin B\}$, that is, the set of all elements of A that are not in B.

a) $(1, \infty) \cap [0, 2]$

b) $(1, 3) \cup [0, 2]$

c) $([3, 5] \cup [0, 4]) \setminus [2, 6]$

d) $[3, 5] \cup ([0, 4] \setminus [2, 6])$

2. Determine the images of the following functions

a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$

b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 1$

c) $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \ln x$

d) $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \dfrac{1}{1 + x}$. Some justification of your answer is required.

3. Determine which of the maps in question 2 is

(i) injective

(ii) surjective

(iii) bijective

4. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

a) Show that if f and g are surjective then $g \circ f : X \to Z$ is surjective

b) Show that if $g \circ f$ is surjective then g is surjective.

c) Give an example to show that it is possible for $g \circ f$ to be surjective even if f is not.

5. Give a conditional definition of the following sets. If you are not sure of the answer, partial credit will be given for correct reasoning, even if the answer is wrong.

a) $\{5n + 1 : n \in \mathbb{Z}\}$

b) $\{x^2 - 1 : x \in \mathbb{R}\}$

c) $\left\{ \dfrac{x}{x + 1} : x \in (0, \infty) \right\}$

6. Give a constructional definition of the following sets

a) $\{n \in \mathbb{Z}_+ : 2 \not| n\}$

b) $\{x \in \mathbb{R} : x \leq 0\}$

c) $\{n \in \mathbb{Z}_+ : p \not| n \text{ for any prime } p \geq 3\}$

I will collect solutions at the lecture on Wednesday 12th November. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 (the Pure Mathematics divisional office) will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

7. Write the following sets as simply as possible in terms of intervals. If you are not sure of the answer, partial credit will be given for correct reasoning, even if the answer is wrong. Note that $A \setminus B$ means $\{x \in A : x \notin B\}$, that is, the set of all elements of A that are not in B.

a) $[1, 3) \cap (2, 4]$

b) $(1, 3) \cup [2, 4]$

c) $([2, 5] \cup [1, 4]) \setminus (0, 3)$

d) $([2, 5] \cup [1, 4]) \setminus (3, 4)$

8. Determine the images of the following functions

a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^4$

b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x - 2$

c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$

d) $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = 1 + \dfrac{1}{x}$.

9. Determine which of the maps in question 7 is

(i) injective

(ii) surjective

(iii) bijective

10. Give a conditional definition of the following sets. If you are not sure of the answer, partial credit wil be given for correct reasoning, even if the answer is wrong.

a) $\{3n - 1 : n \in \mathbb{Z}\}$

b) $\{\sin^2 x : x \in \mathbb{R}\}$

c) $\left\{\dfrac{x}{x+2} : x \in (0, \infty)\right\}$

11. Give a constructional definition of the following sets

a) $\{n \in \mathbb{Z}_+ : 100|n\}$

b) $\{x \in \mathbb{R} : x \geq 1\}$

You might like to try this if you have time, or keep it for later, maybe for revision.

12. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions

a) Show that if f is increasing and g is increasing then $f \circ g$ is increasing.

b) Show that if f is strictly increasing and g is strictly decreasing then both $f \circ g$ and $g \circ f$ are strictly decreasing.

c) Given an example to show that if both f and g are strictly increasing, then the product $f \cdot g$ might be neither increasing nor decreasing.