MATH105 Problem Sheet 4: Integers
Due Wednesday 29th October

1. For each of the pairs of integers m and n given below, use the Euclidean algorithm to find:

(i) the greatest common divisor (g.c.d.) d,

(ii) integers a and b such that $d = am + bn$,

(iii) integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$.

a) $m = 28$, $n = 105$.

b) $m = 213$, $n = 741$.

c) $m = 3146$, $n = 2783$.

2. For each of the pairs of integers m and n given below, find the g.c.d and l.c.m, first by using the Euclidean algorithm and then by using prime factorisation.

a) $m = 77$, $n = 21$

b) $m = 357$, $n = 119$

c) $m = 2898$, $n = 1495$

3. Factorise the polynomial $x^3 + 1$ as a product of a degree 1 and degree 2 polynomial. Use this to help find all prime divisors of $n^3 + 1$ for all integers n with $2 \leq n \leq 10$.

4. Let $a_n \cdots a_1$ denote the usual 10-ary expansion of a natural number, that is,

$$a_n \cdots a_1 = \sum_{k=1}^{n} a_k 10^{k-1}.$$

a) Show that

$$x^m - 1 = (x - 1) \cdot \sum_{i=0}^{m-1} x^i$$

for $m \geq 1$ and hence or otherwise, show that $9 | 10^m - 1$ for all integers $m \geq 0$. This fact is used in proving the test for divisibility by 9:

$$9 | a_n a_{n-1} \cdots a_1 \Leftrightarrow 9 | \sum_{k=1}^{n} a_k$$

You are not asked to prove this but verify that the test works for 216 and 361125, that is, that the digit sums are divisible by 9 and that the numbers themselves are divisible by 9.

b) Using the factorisation of $x^3 + 1$ from question 3 or otherwise, show that $13 | 10^3 + 1$ and $10^6 - 1$. This fact is used in proving the test for divisibility by 13:

$$13 | a_n a_{n-1} \cdots a_1 \Leftrightarrow 13 | (a_1 - a_4 + a_7 \cdots) - 3(a_2 - a_5 + a_8 \cdots) - 4(a_3 - a_6 + a_9 \cdots)$$

You are not asked to prove this but verify that the test works for 1495 and 74802.

I will collect solutions at the lecture on Wednesday 29th October. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

5. For each of the pairs of integers m and n given below, use the Euclidean algorithm to find:

(i) the greatest common divisor (g.c.d.) d,

(ii) integers a and b such that $d = am + bn$,

(iii) integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$.

Also find the g.c.d. and l.c.m. by prime factorisation.

a) $m = 168$, $n = 63$.

b) $m = 234$, $n = 416$.

c) $m = 543$, $n = 1251$.

6. Factorise the polynomial $x^4 - 1$ as a product of two polynomials of degree one and one of degree two. Use this to help factorise $100^4 - 1$ as a product of primes. Also, use the factorisation to show that 8 divides $x^4 - 1$ whenever x is an odd integer.
Hint: Show that one of $x - 1$ or $x + 1$ is divisible by 4, if x is odd.

Anyone who has time might like to try the following. Warning: it is quite a bit harder.

7. Let u_n be the *Fibonacci sequence*, that is, $u_1 = u_2 = 1$ and $u_{n+1} = u_{n-1} + u_n$ for all $n \geq 2$.

a) Prove by induction *Binet's Formula*, that, for all $n \geq 1$,

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Hint Because u_{n+1} is defined in terms of u_n and u_{n-1}, the cases $n = 1$ and $n = 2$ are both **base cases** and you will need to make the theorem to be proved:

For $1 \leq k \leq n$,

$$u_k = \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}.$$

That is, the inductive step is to show that, for $n \geq 2$, **if** this is true for n **then** it is true for $n + 1$.

b) Compute u_n for $n \leq 12$ and $(1+\sqrt{5})^n/(2^n\sqrt{5})$ for $n \leq 12$. Do you notice anything, and if so can you explain it?