MATH105 Problem Sheet 3: Integers
Due Wednesday 22nd October

1. Let a_n be defined by $a_0 = 3$ and $a_{n+1} = 2a_n - 1$ for all $n \in \mathbb{N}$. Prove by induction that $a_n = 2^{n+1} + 1$ for all $n \in \mathbb{N}$.

2. This problem comes from Peter Eccles' book. For $n \in \mathbb{Z}_+$, the number a_n is defined inductively by

$$a_1 = 1,$$
$$a_{n+1} = \frac{6a_n + 5}{a_n + 2}, \ n \in \mathbb{Z}_+.$$

Prove by induction on $n \in \mathbb{Z}_+$ that

(i) $a_n > 0$,

(ii) $a_n < 5$.

3. Prove by induction that $3^n < n!$ for all $n \in \mathbb{N}$ with $n \geq 7$.

4. Find all the divisors of

a) 104

b) 462

c) 3432

Caution: the last of these has 32 divisors

5. Prove by induction on integers $n \geq 2$ that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}.$$

I will collect solutions at the lecture on Wednesday 22nd October. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

6. Prove by induction on n that $2^n + n^2 < 3^n$ for all $n \in N$ with $n \geq 2$. You might find it helpful to use $(n+1)^2 \leq \frac{9}{4}n^2$ for $n \geq 2$.

7. Prove by induction on n that if a_n is defined inductively for $n \in \mathbb{N}$ by $a_0 = 2$ and $a_{n+1} = 3a_n - 2$ then $a_n = 3^n + 1$ for all $n \in \mathbb{N}$.

8. Prove by induction on n that if $a_0 = 1$ and a_n is defined inductively for $n \in \mathbb{N}$ by $a_{n+1} = \dfrac{a_n + 1}{3a_n + 1}$ then $\dfrac{1}{2} \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

Anyone who has time might like to try this. Or you might like to use it for practice at a later date. This is an induction exercise, but rather different from the others because it is a proof of the distributivity law.

9. Starting from the inductive definition of multiplication in $\mathbb{Z}_+$ given by

$$m \cdot 1 = m, \quad m \cdot (n+1) = (m \cdot n) + m,$$

prove by induction on $p \in \mathbb{Z}_+$ that

$$m \cdot (n+p) = (m \cdot n) + (m \cdot p)$$

for all m, n, $p \in \mathbb{Z}_+$. You may assume *associativity of addition for* $\mathbb{Z}_+$, that is

$$(k_1 + k_2) + k_3 = k_1 + (k_2 + k_3) \text{ for all } k_1,\ k_2,\ k_3 \in \mathbb{Z}_+.$$