MATH105 Problem Sheet 2: Standard notation and induction

Due Wednesday 15th October

1. State whether or not the following element x is in the set X.

a) $x = -1$, $X = \mathbb{N}$.

b) $x = -1$, $X = \mathbb{Z}$.

c) $x = -1$, $X = (0, \infty)$.

d) $x = -1$, $X = (-4, -1]$.

e) $x = -1$, $X = (-1, \infty)$.

2. Determine whether the following statements are true or false. Give a brief reason in each case.

a) $\forall n \in \mathbb{N}, \quad n \geq 1$.

b) $\exists n \in \mathbb{N}$ such that $n \leq 1$.

c) $\forall n,\ m \in \mathbb{N}$ with $n < m$, $\exists p \in \mathbb{N}$ such that $n < p < m$.

d) $\forall x,\ y \in \mathbb{Q}$ with $x < y$, $\exists z \in \mathbb{Q}$ such that $x < z < y$.

e) $n \in \mathbb{N} \Rightarrow n \geq 1$.

3. Negate statements a), b) and e) in question 2, using logical symbols where possible. (You are not asked whether these statements are true or false because your answers to question 2 already give this: the negation of a true statement is false and the negation of a false statement is true.)

4. Find the set of all x such that the following inequalities hold. Use implication signs correctly and express answers in the form of intervals or unions of intervals.

a) $x^2 \geq 16$

b) $x^2 + 5x - 6 \geq 0$

c) $\left|\dfrac{3x+2}{2x+3}\right| < 1$

5. Prove by induction that $n^2 < 3^n$ for all $n \in \mathbb{N}$ with $n \geq 2$. Verify separately that $n^2 < 3^n$ also holds for $n = 0$ and $n = 1$

I will collect solutions at the lecture on Wednesday 15th October. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

6. State whether or not the following element x is in the set X.

a) $x = 1.5$, $X = \mathbb{Z}$

b) $x = 1.5$, $X = \mathbb{R}$

c) $x = 1.5$, $X = (1, 2)$.

7. Determine whether the following statements are true or false. Give a brief reason in each case.

a) $\forall n \in \mathbb{N}$, $n > 0$

b) $\exists n \in \mathbb{N}$ such that $n \leq 0$.

c) $\forall n \in N$, $\exists m \in N$ such that $n < m$.

d) $\forall x \in \mathbb{Q}$, with $x > 0$, $\exists y \in \mathbb{Q}$ such that $0 < y < x$.

8. There are two statements in the question above which are the negations of each other. Which two?

9. Find the set of all x such that the following inequalities hold. Use implication signs correctly and express answers in the form of intervals or unions of intervals.

a) $x^2 \leq 2.25$

b) $x^2 - 3x - 4 \geq 0$

c) $\left|\dfrac{3x-1}{x+2}\right| < 1$

10. Prove by induction on n that $2n + 1 \leq 2n^2$ for all $n \in \mathbb{N}$ with $n \geq 2$.

General Remarks on Induction

Two correct proofs by induction may look a bit different but *the key elements have to be present.* To illustrate this let's take a theorem

$$a_n > 0 \;\; \forall \;\; n \in \mathbb{N}, \quad n \geq n_0$$

that we want to prove. This is is just an example but whatever we want to prove is really a theorem which has to be proved for each integer $n \geq n_0$. Here n_0 can be any natural number. In the exercises set it was 2, 7, 0, 1 or 2. It does not matter what a_n is. It could be the a_n used in question 6, just as an example. A proof by induction goes:

a) Prove the theorem for $n = n_0$. (This is the **base case**.)

b) Prove that for $n \geq n_0$, we have $a_n > 0 \Rightarrow a_{n+1} > 0$. (This is the **inductive step**.)

c) Note that "by induction" it follows that $a_n > 0$ for all $n \in \mathbb{N}$ with $n \geq n_0$.

Here are some remarks on these.

(i) It does not matter if the base case is proved before or after the inductive step, but both of these must be done. Usually the base case is pretty easy. Sometimes it is so easy that it can be missed.

(ii) The inductive step is that *if* $a_n > 0$ *then* $a_{n+1} > 0$. It is **not** correct to write down $a_{n+1} > 0$ at the start of the proof. I call this *writing backwards.* Please write forwards. It is much clearer. It will come with practice. If you have trouble with it then try working in rough first.