MATH105 Problem Sheet 8: Equivalence relations and rational numbers

This homework is not for handing in. Please try all the questions on the sheet. It would be a good idea if you tried to do questions 1 to 4 without the help provided by them. Please write out your answers to questions 1 to 4 properly and keep them safe. Writing clearly and correctly is an important part of doing mathematics. Solutions to all problems will be provided the following week.

1. Determine which of the following relations $\sim$ are equivalence relations on the set X. For any which are equivalence relations, determine all the equivalence classes.

a) $X = \mathbb{Z}$, and $\sim$ is defined by $m \sim n \Leftrightarrow 10|(m - n)$.

b) $X = \mathbb{Z}$, and $m \sim n \Leftrightarrow mn = 1$.

c) $X = \mathbb{Z}$ and $m \sim n \Leftrightarrow$ there is an integer $k \geq 2$ *depending on* m *and* n such that $k^2|(m - n)$.

2. Let Y be a nonempty finite set and let $X = 2^Y$, the set of subsets of Y. Let $\sim$ be defined on X by $A \sim B \Leftrightarrow |A| = |B|$

a) Show that $\sim$ is an equivalence relation on X.

b) In the special case $Y = \{1, 2, 3, 4\}$, write down all the equivalence classes of $\sim$.

c) In the general case $|Y| = n$ for $n \in \mathbb{Z}_+$, determine how many equivalence classes of $\sim$ there are, and how many elements of X there are in each equivalence class. (*Caution*: the number of elements in different equivalence classes is not usually the same.)

3.

a) Find the least possible $n \in \mathbb{Z}_+$ such that $\{\frac{r}{n} : r \in \mathbb{Z}\}$ contains $\{\frac{k}{m} : k \in \mathbb{Z}, m \in \{4, 15, 35\}\}$.

b) For n as in a), find a, b, $c \in \mathbb{Z}$ where

$$\frac{1}{n} = \frac{a}{4} + \frac{b}{15} + \frac{c}{35}.$$

Hint. Write $n = 4 \times n_1 = 15 \times n_2 = 35 \times n_3$ and then find a, b, $c \in \mathbb{Z}$ such that $1 = an_1 + bn_2 + cn_3$. Since $n_1 = dm_1$ and $n_2 = dm_2$ where d is the g.c.d. of n_1 and n_2 and d and n_3 are coprime, this is equivalent to finding a_1, b_1, a_2 and $b_2 \in \mathbb{Z}$ such that

$$a_1m_1 + b_1m_2 = a_2d + b_2n_3 = 1.$$

Since m_1, m_2 and n_3 are not large, you can probably do this by inspection.

4. Assuming that $\sqrt{2}$ is not rational, show that:

a) $\dfrac{1}{3} + \dfrac{2}{5}\sqrt{2}$ is not rational;

b) $a + b\sqrt{2}$ is not rational, for any a, $b \in \mathbb{Q}$ with $b \neq 0$;

c) $2^{1/4}$ is not rational, where this is defined to be the real number $x > 0$ with $x^4 = 2$.

5. Determine which of the following relations $\sim$ are equivalence relations on the set X.

a) $X = \mathbb{R}$, and $\sim$ is defined by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

b) $X = \mathbb{R}$, and $x \sim y \Leftrightarrow x - y \in \mathbb{N}$.

c) $X = \mathbb{Z}_+$ and $m \sim n \Leftrightarrow m/n \in \mathbb{Z}_+$

6. Let X denote the set of all vectors (m, n) with m, $n \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation where $\sim$ is defined by

$$(m_1, n_1) \sim (m_2, n_2) \ \Leftrightarrow m_1 - m_2 \in \mathbb{Z} \wedge n_1 - n_2 \in \mathbb{Z}.$$

Write down a representative of each equivalence class. (There are only finitely many.)

7.

a) Find the least possible $n \in \mathbb{Z}_+$ such that $\{\frac{r}{n} : r \in \mathbb{Z}\}$ contains $\{\frac{k}{m} : k \in \mathbb{Z}, m \in \{11, 15, 35\}\}$.

b) For n as in a), find a, b, $c \in \mathbb{Z}$ where

$$\frac{1}{n} = \frac{a}{11} + \frac{b}{15} + \frac{c}{35}.$$

Hint. $n = l.c.m(11, n_1)$ where $n_1 = l.c.m.(15, 35)$. So one possibility is to find b_1 and c_1 such that

$$\frac{b_1}{15} + \frac{c_1}{35} = \frac{1}{n_1}$$

and then find a and e such that

$$\frac{a}{11} + \frac{e}{n_1} = \frac{1}{n}.$$

Then we can take $b = b_1 e$ and $c = c_1 e$.

8. Assuming that $\sqrt{3}$ is not rational, show that:

a) $\frac{1}{7} + \frac{3}{4}\sqrt{3}$ is not rational;

b) $a + b\sqrt{3}$ is not rational, for any a, $b \in \mathbb{Q}$ with $b \neq 0$;

c) $3^{1/6}$ is not rational, where this is defined to be the real number $x > 0$ with $x^4 = 3$.

9. Let x_n be defined for $n \in \mathbb{N}$ by $x_0 = x_1 = 2$ and $x_{n+1} = 2x_n + 3x_{n-1}$ for $n \geq 1$. Prove by induction on n that $x_n = 3^n + (-1)^n$ for all $n \in \mathbb{N}$.
Hint. This is one of the ones where there are two base cases to verify, for $n = 0$ and $n = 1$, and then the inductive hypothesis is that $x_k = 3^k + (-1)^k$ for all $k \leq n$. This is because this is needed for at least $k = n - 1$ and $k = n$ in order to prove it for $k = n + 1$.