The proof of the following is from wikipedia on 9 December 2010. See
http://en.wikipedia.org/wiki/Transcendental_number
and go to section 4 of that article.

Theorem 1. *e is transcendental*

Proof. Suppose for contradiction that e is algebraic. This means that there is n and there are integers a_j for $0 \le j \le n$ such that

$$\sum_{j=0}^{n} a_j e^j = 0, \tag{1}$$

with $a_0 \neq 0$ and $a_n \neq 0$. Now we multiply the equation (1) by I where I is an integral:

$$I = \int_0^\infty p(x)e^{-x}dx$$

where $p(x)$ is a carefully chosen polynomial with integer coefficients. This gives an equation

$$\sum_{j=0}^{n} a_j e^j \int_0^\infty p(x)e^{-x}dx = 0 \tag{2}$$

This can be rewritten, by splitting up the integral in different ways, as

$$\sum_{j=0}^{n} a_j e^j \int_j^\infty p(x)e^{-x}dx = -\sum_{j=1}^{n} a_j e^j \int_0^j p(x)e^{-x}dx \tag{3}$$

The idea is then to show that the right-hand side of (3) is much smaller than the left-hand side, and so they cannot be equal, which is a contradiction.

The key to the whole argument is the fact that, for any natural number m,

$$\int_0^\infty x^m e^{-x}dx = m!.$$

This can be proved by induction, starting from the base case $m = 0$.

The choice for $p(x)$ is

$$p(x) = x^k \prod_{j=1}^{n} (j-x)^{k+1}$$

where k is yet to be chosen. Note that the lowest power of x in $p(x)$ is x^k. In fact

$$p(x) = (n!)^{k+1}x^k + \sum_{i=k+1}^{k+n+nk} b_i x^i$$

for some integers b_i. It follows that

$$\int_0^\infty p(x)e^{-x}dx = (n!)^{k+1}k! + c_0(k+1)!$$

for some integer c_0.

Now we consider the other terms on the left-hand side of (3). If j is an integer with $1 \leq j \leq n$ then

$$e^j \int_j^\infty p(x)e^{-x}dx = \int_j^\infty p(x)e^{-(x-j)}dx = \int_0^\infty p(t+j)e^{-t}dt$$

But

$$p(t+j) = (t+j)^k \prod_{i=1}^{j-1}(i-j-t)^{k+1}(-t)^{k+1} \prod_{i=j+1}^{n}(i-j-t)^{k+1}$$

which is a polynomial in which the lowest power of t is t^{k+1}. So, for $1 \leq j \leq n$

$$\int_j^\infty p(t+j)e^{-t}dt = c_j(k+1)!$$

for an integer c_j. So equation (3), when divided by $k!$, becomes

$$a_0(n!)^{k+1} + (k+1)\sum_{j=0}^{n} a_j c_j = -\sum_{j=1}^{n} \frac{a_j e^j}{k!} \int_0^j p(x)e^{-x}dx. \tag{4}$$

The left-hand side is an integer which can be made non-zero by choice of k. If we choose $k+1$ to be a prime which is bigger than both n and a_0 then the left-hand side of (4) is an integer which is not divisible by $k+1$ and so cannot be 0. So then it suffices to show that the right-hand side of (4) is less than 1 in modulus, if k is sufficiently large. To see this we note that if $0 \leq x \leq n$ then

$$|p(x)| \leq n^k \times n^{n(k+1)} = n^n \times (n^{n+1})^k.$$

Hence, for $1 \leq j \leq n$,

$$\left| \frac{1}{k!} \int_0^j p(x)e^{-x}dx \right| \leq n^{n+1} \frac{(n^{n+1})^k}{k!}$$

This tends to 0 as $k \to \infty$. So the right-hand side of (4) is less than 1 in modulus if k is sufficiently large. This gives the required contradiction. $\square$