MATH104 Exam September 2008, Solutions

All questions except q.1 are standard homework examples

1. lambda, pi, α, μ. (1 mark each.)

2.

(a) $f(n) = 2n + 7$. (3 marks.)

(b) $f(n) = 10^{n+1}$. (4 marks.)

(c)

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

(5 marks.)

3.

a) $x^2 < 1$. (2 marks.)

b) $0 \leq x < 1$. (The answer '$x \geq 0$ and $x < 1$' is also acceptable.) (3 marks.)

c) There exist x, y with $x > y$ and $f(x) \geq f(y)$. (4 marks.)

d) $\forall M \in \mathbb{N}$, $\exists x \in \mathbb{R}$, $f(x) > M$. (3 marks.)

4.

(a) $f(x) = 5 - 6x$ is injective. For let x and y be any real numbers. Then

$$\begin{aligned} f(x) = f(y) \quad &\Longrightarrow \quad 5 - 6x = 5 - 6y \\ &\Longrightarrow \quad -6x = -6y \\ &\Longrightarrow \quad x = y. \end{aligned}$$

(4 marks.)

(b) $f(x) = x^2 + 1$ is not injective. For $f(-1) = f(1)$, but $-1 \neq 1$. (4 marks.)

(c) $f(x) = x^2 + 1, x \geq 0$ is injective. For $f(x) = f(y)$ implies $x^2 = y^2$ and since x and y are ≥ 0 this gives $x = y$. (Or: $(x - y)(x + y) = 0$, and $x + y = 0$ only when $x = 0, y = 0$, so $x = y$ in any case.) (6 marks.)

5.

S not closed under addition means that there exist m, n in S such that $m + n$ is not in S. (2 marks.)

(a) S is closed under addition. For let m and n be any elements of S. Then there exist $k, \ell \in \mathbb{Z}$ such that $m = 3k$ and $n = 3\ell$. Then $m + n = 3(k + \ell) \in S$, since $k + \ell \in \mathbb{Z}$. (4 marks.)

(b) S is not closed under addition. For $m = 30 \in S$ and $n = 31 \in S$, but $m + n = 61 \notin S$. (4 marks.)

(c) S is closed under addition. For let m and n be any elements of S. Then $m \geq 50$ and $n \geq 50$, so $m + n \geq 100 \geq 50$. (4 marks.)

6.

Context X is a metric space. (1 mark.)

Hypothesis X is complete and X is totally bounded. (1 mark.)

Conclusion X is compact. (1 mark.)

Contrapositive *Let X be a metric space. If X is not compact, then X is not complete or X is not totally bounded.* (2 marks.)

(a) Nothing. (2 marks.)

(b) X is not complete or X is not totally bounded. (2 marks.)

(c) X is not totally bounded. (3 marks.)

(d) Nothing. (2 marks.)

7.

(a) Let $m, n \in \mathbb{Z}$, and suppose that $7|m$ and $7|n$. Then there exist integers k and ℓ such that $m = 7k$ and $n = 7\ell$. Hence $m + n = 7(k + \ell)$ is divisible by 7 as required. (5 marks.)

(b) Let $a, b, c \in \mathbb{Z}$, and suppose that $a|b$ and $b|c$. Then there exist integers k and ℓ such that $b = ka$ and $c = \ell b$. Hence $c = (\ell k)a$, so that $a|c$ as required. (5 marks.)

(c) Let $m, n \in \mathbb{Z}$ and suppose that $7|m$ and $7 \nmid (m+n)$. Assume for a contradiction that $7|n$. Then by part (a), $7|(m + n)$. However $7 \nmid (m + n)$. This is the required contradiction.(5 marks.)

8.

(a) False. (5 is not a perfect square, or: $n^2 \leq 4$ when $|n| \leq 2$, and $n^2 \geq 9$ when $|n| \geq 3$.) (3 marks.)

(b) False. ($n = 0$ is a counterexample.) (3 marks.)

(c) True. (Given any real number x, take $y = x/3$.) (3 marks.)

(d) False. ($x = 2$.) (3 marks.)

(e) False. (Given any $x \in \mathbb{R}$ take $y = x - 2$.) (3 marks.)