MATH104 Exam May 2009, Solutions

All questions except q.1 are standard examples similar to classwork or homework.

1. delta, zeta ϵ, ω. (1 mark each.)
$1 + 1 + 1 + 1 = 4$ marks

2.

(a) $f(n) = -3n$.

(b) $f(n) = n(n + 1)$

(c) $\{-1, 1\}$.

(d) $1 \leq x$

(2 marks for (a), 4 marks for (b) and 3 marks for each of c) and d).)
$2 + 4 + 3 + 3 = 12$ marks

3. The negations are as follows

(a) $x \leq 10$ or $x \geq 11$ (1 mark)

(b) There exists $x \in \mathbb{R}$ such that $x^2 - 1 = 0$ and $x \neq 1$.(2 marks)

(c) For all $x \in \mathbb{R}$ with $x \geq 2$, we have $f(x) < 1$(2 marks)

(d) There exists a function $f(x)$which is both even and odd. (2 marks)

For the original statements (or equivalently, for their negations),in a), x is a free variable, and in c) the function $f(x)$ is a free variable (4 marks)

b) is false and its negation is true, taking $x = -1$ (2 marks)

d) is false and its negation is true, taking $f(x) = 0$ for all x (2 marks)

$1 + 2 + 2 + 2 + 4 + 2 + 2 = 15$ marks

4.

(a) f is *not strictly increasing* if there exist x, y in the domain of f such that $x < y$ but $f(x) \geq f(y)$. (2 marks)

(b) $f(1) = |1| = |-1| = f(-1)$, and so f is not injective.(2 marks)

(c) $f(0) = 0$ and $f(2) = -2$. So f is not strictly increasing. (2 marks)

To show that f is injective:

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x(1-y) = y(1-x) \Rightarrow x - xy = y - xy \Rightarrow x = y. \text{ (4 marks)}$$

(d) It suffices to show that if $x \neq y$ then $f(x) \neq f(y)$. If $x \neq y$ then either $x < y$ or $y < x$. We can assume that $x < y$, renaming if necessary. Then $f(x) < f(y)$ and so $f(x) \neq f(y)$ (5 marks)

$2 + 2 + 2 + 4 + 5 = 15$ marks

5.
a) S closed under addition means that, for all $x \in S$ and $y \in S$, it follows that $x + y \in S$. (1 mark)

$1 = 1^2 \in S$ and $4 = 2^2 \in S$ but $4 + 1 = 5$ is not of the form n^2 for any $n \in S$ because $4 = 2^2 < 5 < 3^2 = 9$. So S is not closed under addition. (2 marks)
b) $m|n$ means that there exists an integer k with $n = km$. (1 mark)

(i) From the definition, $n \in S \Leftrightarrow n = 5k$ for $k \in \mathbb{Z}$. So if $n_1 \in S$ and $n_2 \in S$, we can write $n_1 = 5k_1$ and $n_2 = 5k_2$ for k_1, $k_2 \in \mathbb{Z}$, and $n_1 + n_2 = 5(k_1 + k_2) \in S$. So S is closed under addition. (2 marks)

(ii) $n \in S \Leftrightarrow n = 5k$ for $k \in \mathbb{Z} \Rightarrow n^2 = 25k^2 = 5 \cdot (5k^2) \Rightarrow n^2 \in S$. (2 marks)

(iii) $n \pm 1 \in S \Rightarrow n = 5k \mp 1$ with $k \in \mathbb{Z} \Rightarrow n^2 = (5k \mp 1)^2 = 25k^2 \mp 10k + 1 = 5 \cdot (5k^2 \mp 2k) + 1 \Rightarrow n^2 - 1 \in S$. (4 marks)

(iv) Similarly $n \pm 2 \in S \Rightarrow n = 5k \mp 2$ with $k \in \mathbb{Z} \Rightarrow n^2 = (5k \mp 2)^2 = 25k^2 \mp 20k + 4 = 5 \cdot (5k^2 \mp 4k + 1) - 1 \Rightarrow n^2 + 1 \in S$. (3 marks)

$1 + 2 + 1 + 2 + 2 + 4 + 3 = 15$ marks

6.

a) R is not an equivalence relation. Property (i) fails, since for example $1\,R\,1$ is false. (2 marks)

b) R is an equivalence relation. For let x, y, and $z \in \mathbb{R}$

(i) $|x| = |x|$ and so $x\,R\,x$.
(ii) If $x\,R\,y$ then $|x| = |y|$, and so $|y| = |x|$ and $y\,R\,x$.
(iii) If $x\,R\,y$ and $y\,R\,z$ then $|x| = |y|$ and $|y| = |z|$, and so $x\,R\,z$.

(4 marks)

c) R is an equivalence relation. For let x, y, and $z \in \mathbb{R}$

$x - x = 0 \in \mathbb{Z}$. So $x\,R\,x$.
If $x - y = n \in \mathbb{Z}$ then $y - x = -n \in \mathbb{Z}$. So if $x\,R\,y$ then $y\,R\,x$
If $x\,R\,y$ and $y\,R\,z$ then $x - y = n \in \mathbb{Z}$ and $y - z = m \in \mathbb{Z}$, and so then $x - z = n + m \in \mathbb{Z}$ and $x\,R\,z$.

(4 marks)

$2 + 4 + 4 = 10$ marks

7.

Context X is a Banach space and $T : X \to X$ is a linear map. (1 mark)

Hypothesis T is bounded (1 mark)

Conclusion The spectrum of T is a closed bounded non-empty subset of $\mathbb{C}$. (1 mark)

Converse *Let the spectrum of T be a closed bounded non-empty subset of $\mathbb{C}$. Then T is bounded.* (2 marks)

(a) Nothing: hypothesis of theorem not fully satisfied. (2 marks)

(b) T is not bounded. Since the conclusion of the theorem does not hold, the hypothesis cannot hold fully. (2 marks)

(c) Nothing can be deduced since there is no reason for the converse of the theorem to be true(2 marks)

(d) These are the hypotheses of the original theorem. Therefore the conclusion must hold and the spectrum of T is a closed non-empty bounded subset of $\mathbb{C}$. (2 marks)

However if the converse of the theorem is true then we can say nothing about (b) because if the hypothesis of the converse theorem does not hold the conclusion of the converse (that is the hypothesis of the original theorem) may or may not hold. (2 marks)
$1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 = 15$ marks

8.

a) False. ($x = 0$ gives $x^2 - 1 < 0$.) (2 marks)

b) True. (Take $x = \frac{1}{2}$ for example.) (3 marks)

c) True. (Take $a = 1$ and $b = 2$ for example. Then $(a + b)^2 = 9$ and $5ab = 10$) (2 marks)

d) True. (Simply true. For any $x > 0$ we can find a positive integer n with $n > \frac{1}{x}$) (3 marks)

e) False. Fix any $n \in \mathbb{N}$. If $n = 0$ take $x = 1$ and if $n > 0$ take $x = \frac{1}{n}$. In both cases it is *false* that $\frac{1}{x} < n$. (4 marks.)

$2 + 3 + 2 + 3 + 4 = 14$ marks