Notes for 21-24 January 2008: Taylor Polynomials and Taylor Series

Suppose that, for all x,

$$f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$$

Putting $x = a$,

$$f(a) = a_0.$$

Differentiating,

$$f'(x) = a_1 + 2a_2(x-a) + \cdots + na_n(x-a)^{n-1}.$$

Putting $x = a$,

$$f'(a) = a_1.$$

Differentiating,

$$f''(x) = 2a_2 + 6a_3(x-a) + \cdots + n(n-1)a_n(x-a)^{n-2}.$$

Putting $x = a$,

$$\frac{f''(a)}{2} = a_2.$$

Differentiating,

$$\begin{aligned} f^{(3)}(x) &= 6a_3 + (4 \times 3 \times 2)a_4(x-a) \\ &+ \cdots + n(n-1)(n-2)a_n(x-a)^{n-3}. \end{aligned}$$

Putting $x = a$,

$$\frac{f^{(3)}(a)}{3!} = a_3.$$

...In general,

$$\frac{f^{(k)}(a)}{k!} = a_k,$$

and so assuming that f is a polynomial of degree $\leq n$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

So it is natural to assume, under suitable conditions, that for a general function f which is n-times differentiable, for x near a,

$$f(x) \approx f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The first approximation, for $n = 1$, is certainly true, because by the definition of derivative, for x near a,

$$f(x) \approx f(a) + f'(a)(x-a).$$

The *n'th Taylor polynomial* $P_n(x)$ *of* f *at* a (or more correctly, $P_n(x,a)$) is given by

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

$$= f(a) + f'(a)(x-a) + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The *Taylor series of* f *at* a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \cdots \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

This is a *formal power series.* It may not converge for any x - although in many cases it does.

The Remainder Term

For any function f which is $n+1$-times continuously differentiable on the closed interval between x and a,

$$f(x) = P_n(x,a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt. \tag{1}$$

So

$$f(x) = P_n(x,a) + \frac{f^{(n+1)}(c)}{(n)!}\int_a^x (x-t)^n dt,$$

$$f(x) = P_n(x,a) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a. The term

$$R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is called the *remainder term* and is often written just $R_n(x)$. So

$$f(x) = P_n(x,a) + R_n(x,a)$$

(1) can be proved by induction because by integration by parts

$$\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

$$= -\frac{f^n(a)}{n!}(x-a)^n + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)dt.$$