SECTION A

1. Write down the Taylor series about $x = 2$ for the function

$$f(x) = x^{-1}.$$

State whether this Taylor series converges to $f(x)$ for:

a) $x = 1$, b) $x = 4$. [5 marks]

2. Find the solutions of the following differential equations:

(i) $e^y \dfrac{dy}{dx} = x$ with $y(1) = 0$,

(ii) $x\dfrac{dy}{dx} + 2y = x$ with $y(1) = 0$.

[8 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

with the initial conditions $y(0) = 2$, $y'(0) = 1$.

[5 marks]

4. Show, by taking limits along two different paths to the origin $(0, 0)$, that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2}$$

does not exist.

[4 marks]

5. Work out all first and second partial derivatives of

$$f(x,y) = x^4 - 6x^2y^2 + y^4,$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[5 marks]

6. Suppose that $x = x(t)$, $y = y(t)$, and $z = z(t)$ are functions of t such that

$$x(0) = 2, \quad y(0) = -1, \quad z(0) = 0.$$

Suppose that the derivatives satisfy

$$x'(0) = y'(0) = 1, \quad z'(0) = -1.$$

Then work out

$$\frac{dF}{dt}(0)$$

where $F(t) = f(x(t), y(t), z(t))$, and

$$f(x,y,z) = x^2y + \sin(xyz).$$

[5 marks]

7. Find the gradient vector $\nabla f(1,1,1)$, where

$$f(x,y,z) = \frac{xyz}{x^2+y^2+z^2}.$$

Find also the derivative of f in the direction $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ at the point $(1,1,1)$.
[5 marks]

8. Locate and classify all stationary points of the function

$$f(x,y) = x^2y - 2xy + y^2 - 15y.$$

[8 marks]

9. Find the linear approximation near $(x, y) = (2, 1)$ to the function

$$f(x, y) = \frac{1}{x^2 - y^2}.$$

[4 marks]

10. By changing the order of integration, compute

$$\int_0^1 \int_y^1 e^{y/x} dx dy.$$

[6 marks]

SECTION B

11.

In this question, let

$$f(y) = \frac{1}{1+y}, \quad g(x) = \frac{1}{1+x^2}, \quad h(x) = \tan^{-1}(x).$$

(i) Write down the Taylor series of f, g and h, all at 0.

(ii) Show that if $y \geq 0$, the nth remainder term $R_n(y, 0)$ of f at 0 satisfies

$$|R_n(y, 0)| \leq y^{n+1}.$$

Hence show that if $x \geq 0$,

$$\int_0^x R_n(t^2, 0) dt \leq \frac{x^{2n+3}}{2n+3}.$$

(iii) Express $h(1) = \tan^{-1}(1)$ in terms of π. If $P_n(x, 0)$ denotes the nth Taylor polynomial of h at 0, use your calculator to compute $P_{22}(1, 0)$ and $\tan^{-1}(1)$ and verify that

$$\left|\tan^{-1}(1) - P_{22}(1, 0)\right| \leq \frac{1}{23}.$$

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' - 4y' - 5y = 4e^x$$

with $y(0) = 1$, $y'(0) = -1$.

(ii)

$$y'' - 4y' - 5y = -5x^2 + 2x + 5$$

with $y(0) = 1$, $y'(0) = -1$.

[15 marks]

13. Find the maximum and minimum values of the function $f(x, y)$ in the region where $g(x, y) \leq 3$, where $f(x, y)$ and $g(x, y)$ are defined by

$$f(x, y) = xy + x,$$
$$g(x, y) = 3x^2 + y^2.$$

[15 marks]

14.

a) Find the weight of the triangle R bounded by $y = x$, $y = 2x$ and $x = 1$, where the density function is $\rho(x, y) = x$

b) Find the centre of mass $(\overline{x}, \overline{y})$ of R.

[15 marks]