This is supposed to take ten minutes or less.
What is the Taylor series of $f(x) = \ln x$ at 1?

Hint:

We need to calculate all the derivatives of f.

First, at a general point x,

and then at $x = 1$.
So ...

- $f'(x) = x^{-1}$,
- $f''(x) = -x^{-2}$,
- $f^{(3)}(x) = 2x^{-3}$,
- $f^{(4)}(x) = -3!x^{-4}$,
- and the general formula is

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}.$$

So putting $x = 1$ gives:

- $f(1) = \ln(1) = 0$,
- $f'(1) = 1^{-1} = 1$,
- $\dfrac{f''(1)}{2!} = -\dfrac{1}{2}$,
- $\dfrac{f^{(3)}(1)}{3!} = \dfrac{1}{3}$,
- $\dfrac{f^{(4)}(1)}{4!} = -\dfrac{1}{4}$, and in general,

$$\frac{f^{(n)}(1)}{n!} = (-1)^{n-1}\frac{1}{n}.$$

So the Taylor series of $\ln x$ at $x = 1$,

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots$$

$$+\frac{(-1)^{n-1}}{n}(x-1)^n + \cdots$$

is

$$0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$+\frac{(-1)^{n-1}}{n}(x-1)^n + \cdots$$

So the fourth Taylor polynomial $P_4(x)$ of $\ln x$ at 1 is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.$$

The associated remainder term $R_4(x)$ is

$$\frac{f^{(5)}(c)}{5!}(x-1)^5 = \frac{c^{-5}}{5}(x-1)^5$$

for some c between 1 and x.

Now take $x = 2$.

$$f(2) = \ln 2$$

According to the university calculator, this is

$$0.6931..$$

The fourth Taylor polynomial gives the approximation

$$P_4(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} = 0.58333333\cdots.$$

For c between 1 and 2, c^{-5} is largest at $c = 1$. So an upper bound on $|R_5(2)$ is given by

$$|R_5(2)| \leq \frac{1}{5}.$$

So we obtain

$$\left|\ln 2 - \frac{7}{12}\right| \leq 0.2,$$

which is consistent with the calculator's calculation.