An Experiment

28 January 2008

This is supposed to take ten minutes or less.

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and then at $x = 1$.

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$$+\frac{(-1)^{n-1}}{n}(x-1)^n+\cdots$$

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The associated remainder term $R_4(x)$ is

$$\frac{f^{(5)}(c)}{5!}(x-1)^5 = \frac{c^{-5}}{5}(x-1)^5$$

for some c between 1 and x.

Now take $x = 2$.

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$$f(2) = \ln 2$$

According to the university calculator, this is

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The fourth Taylor polynomial gives the approximation

$$P_4(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} = 0.58333333\cdots.$$

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So we obtain

$$\left|\ln 2 - \frac{7}{12}\right| \leq 0.2,$$

which is consistent with the calculator's calculation.