Basics about holomorphic maps

Recall some facts about holomorphic maps. If

$$z = x + iy$$

$$f(z) = u(x, y) + iv(x, y)$$

then

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Considered as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, the derivative is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

If $f'(z) \neq 0$ then $u_x^2 + v_x^2 \neq 0$. Lengths are not usually preserved, but angles are.

The action of the derivative at z_0 is multiplication by $f'(z_0)$. Conversely, suppose that $f : U(\subset \mathbb{R}^2) \to \mathbb{R}^2$ is continuous, and continuously differentiable except at finitely many points, and the dervative Df is invertible, has positive determinant and preserves angles except at finitely many points. Write $f = (u, v)$. The derivative Df is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

If angles are to be preserved then this must be of the form

$$\begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

So the Cauchy-Riemann equations

$$u_x = v_y,$$

$$v_x = -u_y$$

are satisfied, and hence f is holomorphic, except possibly at finitely many points. But since f is continuous, any singularities are removable and f is holomorphic on U.

How to write Riemannian metrics in the plane

The usual classical form of writing a Riemannian metric in the plane is

$$adx^2 + 2bdxdy + cdy^2$$

where a, b, c are real-valued functions of (x, y), and the symmetric matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite. For this we need

$$a + c > 0,$$

$$ac - b^2 > 0.$$

The classical notation is suggested by the formula for the length of a curve $(x(t), y(t))$ $(t \in I)$ in this metric:

$$\int_I \sqrt{a(dx/dt)^2 + 2b(dx/dt)(dy/dt) + c(dy/dt)^2} dt$$

Field of Ellipses

A 2×2 symmetric positive definite matrix A defines an ellipse with equation

$$\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

The constant on the righthand side is unimportant. Note that

$$A = P^T \Delta P,$$

with P orthogonal and Δ diagonal. Interchanging the rows of P if necessary, we can assume that P has determinant 1. Then we get the standard form

$$\begin{pmatrix} X & Y \end{pmatrix} \Delta \begin{pmatrix} X \\ Y \end{pmatrix} = 1$$

for the ellipse by making the change of variable

$$\begin{pmatrix} X \\ Y \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$$

The major and minor axes of the ellipse are orthogonal to each other and are given by the columns of U (not necessarily in that order) provided the eigenvalues of A are distinct.

This association of an ellipse (up to scale) to each point in the domain is called a *field of ellipses*. The major axis at each point — up to direction – gives a *line field*. It is undefined when the eigenvalues of A are equal.

Complex form of a Riemannian metric

In formulating the measurable Riemann mapping theorem it is more convenient to write the metric $adx^2 + 2bdxdy + cdy^2$

in another form:

$$\lambda |dz + \mu d\overline{z}|2 = \lambda |\mu| . |\overline{\mu}^{-1} d\overline{z} + dz|^2$$

where $\lambda > 0$ and $|\mu| < 1$ and λ and $\mu = \mu_1 + i\mu_2$ are functions of z. the function μ is called the *Beltrami differential* (of the Riemannian metric). To get between the two:

$$2\lambda\mu_2 = b,$$

$$\lambda(1 + |\mu|^2 + 2\mu_1) = a,$$
$$\lambda(1 + |\mu|^2 - 2\mu_1) = c.$$

Then
$$ac - b^2 = \lambda^2(1 - |\mu|^2)^2$$
and
$$\frac{ac - b^2}{(a + c)^2} = \frac{1 - |\mu|^2}{1 + |\mu|^2}.$$
So μ is bounded from 1 if the ratio of the eigenvalues of A is bounded above and below, where
$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

There is a relation between the argument of $\mu(z)$ and the major axis of the ellipse associated to the metric at z. If $\pm v$ is the direction of the major axis then
$$\arg(\mu) = \arg(v^{-2}).$$

Transforming Riemannian metrics

If $f : U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^2$, and σ is a Riemannian metric on V then we can define a Riemannian metric $f^*\sigma$ on U by the following formula. If σ is given in classical terminology by $adx^2 + 2bdxdy + cdy^2$ then $f^*\sigma$ is given by
$$\begin{pmatrix} dx & dy \end{pmatrix} Df^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} Df \begin{pmatrix} dx \\ dy \end{pmatrix}$$
where Df is the 2×2 matrix representing the derivative. If $\ell_1(\gamma_1)$ denotes length of a path γ_1 with respect to σ and $\ell_2(\gamma_2)$ denotes length of a path γ_2 with respect to $f^*(\sigma)$ then
$$\ell_2(\gamma) = \ell_1(f \circ \gamma)$$
This follows from the definition of $f^*\sigma$ and the chain rule for differentiating $f \circ \gamma$. Note that f^* is a contravariant functor, that is
$$(f \circ g)^*\sigma = g^* f^* \sigma$$
(where defined).

Transforming the standard metric in the complex notation

The standard metric σ_0 is $dx^2 + dy^2 = |dz|^2$. Suppose that $f : U \to V$ is a diffeomorphism between open subsets U and V of $\mathbb{C}$. So f is a complex-valued function on a complex domain, and the same is true for the partial derivatives f_x and f_y. Write
$$f_z = \frac{1}{2}(f_x - if_y)$$

$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y)$$

If f is holomorphic, then, by the Cauchy-Riemann equations, $f_z = f'$ and $f_{\overline{z}} = 0$. Write

$$dz = dx + idy$$

$$d\overline{z} = dx - idy$$

Then

$$f_x dx + f_y dy = f_z dz + f_{\overline{z}} d\overline{z}$$

Then $f^*\sigma_0$ is given by

$$|f_x dx + f_y dy|^2 = |f_z dz + f_{\overline{z}} d\overline{z}|^2$$

$$= |f_z|^2 \left| dz + \frac{f_{\overline{z}}}{f_z} d\overline{z} \right|^2$$

Transforming fields of ellipses and Beltrami differentials

If σ_0 is the standard metric $dx^2 + dy^2 = |dz|^2$ and g is holomorphic then write

$$f^*\sigma_0 = \lambda_1 |dz + \mu_1 d\overline{z}|^2$$

$$g^* f^* \sigma_0 = \lambda_2 |dz + \mu_2 d\overline{z}|^2$$

Then

$$\mu_2 = \frac{\overline{g(z)}}{g(z)} \mu_1 \circ g$$

$$\lambda_2 = |g'| \lambda_1 \circ g$$

In particular,

$$\|\mu_2\|_\infty = \|\mu_1\|_\infty.$$

Also since

$$D(f \circ g)^T D(f \circ g) = Dg^T (Df^T Df) Dg$$

the major and minor axes for the ellipse at z for $g^* f^* \sigma_0$ map under Dg to those for $f^*\sigma$. If the major axis of the ellipse at z for $g^* f^* \sigma_0$ is in the direction of $\pm v$ ($v \in \mathbb{C}$) then the direction for $f^*\sigma_0$ at $g(z)$ is $\pm g'(z) v$.

The Riemann Mapping Theorem

Write

$$D = \{z : |z| < 1\}$$

The classical Riemann mapping theorem (easy version) says that if U is an simply connected proper open subset of $\mathbb{C}$, then there exists a holomorphic bijection $\varphi : U \to D$. One way to prove this (not the easiest) would be to find an orientation-preserving diffeomorphism $g : D \to U$, giving rise to a Riemannian metric $g^*\sigma_0$ on$D$. As before,

σ_0 denotes the standard metric $|dz|^2$ on U (or on any domain in $\mathbb{C}$). Then suppose we can find an o-p diffeomorphism $f : D \to D$ with

$$f^*\sigma_0 = \lambda g^*\sigma_0$$

for a strictly positive function λ. Then

$$(g^{-1})^* f^*\sigma_0 = (f \circ g^{-1})^*\sigma_0 = \lambda\sigma_0$$

So

$$D(f \circ g^{-1})^T D(f \circ g) = \lambda I$$

Then $D(f \circ g^{-1})$ must be a multiple of an orthogonal matrix and of positve determinant. So the partial derivatives of $f \circ g^{-1}$ satisfy the Cauchy-Riemann equations, and $f \circ g^{-1} : U \to D$ is holomorphic.

The Measurable Riemann Mapping Theorem

This theorem has a long history. The version usually now used is that of L. Ahlfors and L.Bers in *Annals of Math.*, 72 (1960), 385-404. There are versions for $\mathbb{C}$, $\overline{\mathbb{C}}$ and the unit disc D. Let U be any one of these three.

Theorem 1 *Suppose that $\mu \in L^\infty(U)$ with $\|\mu\|_\infty < 1$. Then there exists a homeomorphism $f : U \to U$ which is differentiable a.e., with partial derivatives locally L^p for some $p > 2$ and*

$$\frac{f_{\overline{z}}}{f_z} = \mu$$

That is, for some $\lambda > 0$

$$f^*\sigma_0 = \lambda|dz + \mu d\overline{z}|^2.$$

Moreover f is unique up to left composition with a Möbius transformation.

Such a homeomorphism f is *quasi-conformal* (and o-p). It is holomorphic if $\mu = 0$ a.e.

Quasi-conformal Maps

The standard reference is Ahlfors' book

Lectures on Quasiconformal mappings

Take d to be the Euclidean metric if $D = \mathbb{C}$ or D and the spherical metric if $U = \overline{\mathbb{C}}$. Let $B(z, r)$ denote the ball of radius r centred on z in this metric. The simplest topological definition for a quasiconformal map is the following. $f : U \to U$ is quasiconformal if it is a homeomorphism and there exists a constant K_1 such that for all $z \in U$ and each ball $B(z, r)$, there is r_1 such that

$$B(f(z), r_1) \subset f(B(z, r)) \subset B(f(z), K_1 r_1)$$

Ahlfors gives two definitions which are equivalent to this, and he proves their equivalence, but neither of them is this definition (for good reason).

Modulus of a topological rectangle

Any closed topological disc R in the plane with four marked points x_i ($1 \leq i \leq 4$ in anticlockwise direction) on the boundary is homeomorphic to a rectangle, with the four marked points mapping to the vertices. So R can therefore be referred to as a *topological rectangle*. A strengthening of the Riemann mapping theorem imples that this homeomorphism can be realised by a map which is holomorphic on the interior. For unique numbers $a > 0$, $b > 0$ there is a homeomorphism

$$\varphi : R \to \{x + iy : 0 \leq x \leq a, 0 \leq y \leq b\}$$

whjich is holmorphic on the interior of R and mapping x_1 to 0, x_2 to a, x_3 to $a + ib$, and x_4 to ib. a/b is then defined to be the *modulus* $\text{mod}(R)$ of R.

Ahlfors' definitions

Definition 1 *A homeomorphism $\varphi : U \to U$ is K -quasiconformal if for any topological rectangle R*

$$\frac{\text{mod}(R)}{K} \leq \text{mod}(R) \leq K\text{mod}(R).$$

Definition 2 *A homeomorphism $\varphi : U \to U$ is K -quasiconformal if partial derivatives f_x, f_y exist a.e. in U, and are locally L^1along a.e. horizontal line in U, and a.e. vertical line in U, and*

$$|f_{\overline{z}}| \leq k|f_z|$$

where

$$k = \frac{K-1}{K+1}.$$

Continuity, Differentiability, and Holomorphicity

The Ahlfors Bers paper is famous for results about families of Beltrami differentials which vary continuously, differentiably or holomorphically. We keep to the notation of Theorem 1.

Theorem 2 *Let $\lambda \to \mu_\lambda : \Lambda \to L^\infty(U)$ ($\lambda \in \Lambda$ be a continuous family of Beltrami differentials with $\|\mu_\lambda\|_\infty \leq k$ for some $k < 1$. Then $\lambda \to f_{\mu_\lambda}$ is:*

- *locally uniformly continuous in $C(U)$*
- *locally Hölder on $C^\alpha(U)$ for some $\alpha > 0$*
- *the partial derivatives $(f_{\mu_\lambda})_x$ and $(f_{\mu_\lambda})_y$ are continuous in the local L^p topology.*

If $\lambda \to \mu_\lambda : \Lambda$ is locally uniformly differentiable/holomorphic in L^∞, then $\lambda \to f_{\mu_\lambda}$ is differentiable/holomorphic with respect to the same list of seminorms. In particular this theorem implies that if $\lambda \to \mu_\lambda : \Lambda \to L^\infty(U)$ is continuous/holmorphic, then so is

$$\lambda \to f_{\mu_\lambda}(z) : \Lambda \to U$$

for each $z \in U$.