Representation space

- Throughout this lecture, Γ_0 is a finitely generated group and $\rho : \Gamma_0 \to \Gamma \leq PSL(2, \mathbb{C})$ is a group isomorphism.
- If Γ_0 has a generating set with r elements, then we can identify the set of all (Γ, ρ) with a closed affine subvariety of $(PSL(2, \mathbb{C}))^r$.
- We are interested in the case when Γ is Kleinian, that is discrete.

Quasi-conformal deformations

Definition. (Γ_2, ρ_2) is a *quasi-conformal deformation* of (Γ_1, ρ_1) if there is a quasiconformal homeomorphism φ of $\overline{\mathbb{C}}$ such that $\rho_2(\gamma_0) \circ \varphi = \varphi \circ \rho_1(\gamma_0)$ for all $\gamma_0 \in \Gamma_0$.

- In this case, $\gamma \to \varphi \circ \gamma \circ \varphi^{-1} : \Gamma_1 \to \Gamma_2$ is a group isomorphism.
- The derivative $D\varphi$, which is defined a.e., defines a Γ_1-invariant field of ellipses by
$$\underline{x}^T D\varphi_z^T D\varphi_z \underline{x} = \text{const}.$$
- This also defines a Γ_1- invariant line field, taking the the major axis or 0 depending on whether the ellipse is not, or is, a circle.
- Alternatively, $\varphi_{\overline{z}}/\varphi_z$ is a Γ_1-invariant Beltrami-differential.

Stable representations

Definition.A group Γ, ρ is *stable* if for any representation $\rho : \Gamma_0 \to \Gamma$ and any (Γ', ρ') sufficiently close to (Γ, ρ) there is a homeomorphism $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that
$$\varphi(\rho(\gamma).z) = \rho'(\gamma'.\varphi(z))$$
for all $\gamma \in \Gamma_0$ and $z \in \overline{\mathbb{C}}$. It is relatively straightforward to prove that any finitely generated Kleinian group Γ which acts hyperbolically on L_Γ is stable. The following theorem is due to Sullivan.

Theorem 1. *If Γ is stable then Γ acts hyperbolically on L_Γ.*

The Ingredients

- The Sullivan-Mane-Sad λ-lemma, which implies that all nearby maps are actually quasiconformally conjugate:

  λ**-Lemma** *If $\Lambda \subset \mathbb{C}^n$ is open and $X \subset C$ with $\Phi(\underline{0}, z) = z$ and $\Phi : \Lambda \times X \to \mathbb{C} : (\lambda, z) \mapsto \Phi(\lambda, z)$ is holomorphic in λ, and injective on X for each fixed λ, then the map $z \mapsto \Phi(\lambda, z)$ extends to a quasi-conformal homeomophism from $\overline{X}$ to its image.*
- An argument due to Thurston, which shows that the representation space is bounded below by a sum of numbers, one corresponding to each topological end of the manifold. This, in turn, depends the existence, in hyperbolic 3-manifold with finitely generated fundamental group of the compact *Scott core*;
- The following theorem (also due to Sullivan)

Invariant line fields

Theorem 2. *Let Γ be a finitely generated Kleinian group. Then any Γ-invariant line field is supported a.e. on the domain of discontinuity Ω_Γ.*

The analogues of Sullivan's Theorems for holomorphic maps, even for polynomials, is still unknown, although quasi-conformal rigidity is now known in some cases.

Further remarks.

- The Ahlfors Conjecture, that the limit set of a Kleinian group is either $\overline{\mathbb{C}}$ or of zero measure, has now been proved. This does not imply Sullivan's theorem in the case when the limit set is $\overline{\mathbb{C}}$.
- The analogue of the Ahlfors conjecture is now known to be false for polynomials (Buff and Cheritat).
- An eventual corollary of Sullivan's No-invariant line fields, and the Ahlfors' Finiteness Theorem is that the quasi-conformal deformation space of (Γ, ρ) is a finite-dimensional manifold, whose dimension can be computed.

Recurrent and Dissipative

Definition. The action of Γ on L_Γ is said to be *recurrent* if for any set $U \subset L_\Gamma$ of positive Lebesgue measure, there exists $\gamma \in \Gamma$ with $\gamma \neq I$ such that $U \cap \gamma.U$ has positive measure.

- If the action of Γ on L is not recurrent then there exists a set U of positive measure such that all the sets $\gamma.U$ are disjoint. We then write

$$\Gamma.U = \cup_{\gamma\in\Gamma}\gamma.U$$

- In this case, there is a positive measure set V which is of the form $\Gamma.U$ for some such U, which contains a.e. point of $\Gamma.U_1$, for any measurable set U_1 such that the sets $\gamma.U_1$ are all disjoint.
- Such a set V, which is defined modulo sets of measure 0, is called the *dissipative part* of the action of Γ on L_Γ. the complement is the *recurrent part*

No dissipative part

Sullivan proved the absence of invariant line fields by the following reduction.

Lemma 3. *Let Γ be a finitely generated Kleinian group. Then the action of Γ on L_Γ has no dissipative part modulo sets of measure 0. That is, the action is recurrent.*

Proof

- The proof is by contradiction. We assume that there is a dissipative part.
- This gives an infinite-dimensional space of Γ-invariant Beltrami differentials on $\overline{\mathbb{C}}$.
- This, in turn, gives an infinite-dimensional space of Kleinian groups isomorphic to Γ, which is impossible.

The infinite dimensional space of Beltrami differentials

- Suppose for contradiction that there is a set $U \subset L_\Gamma$ of positive measure such that the sets $\gamma.U$ are all disjoint.
- Then the space of Beltrami differentials supported on U is infinite dimensional. To find an infinite linearly independent set we can for example choose disjoint positive measure sets U_j in U and let $\mu_j \in L^\infty(U_j)$ with $\|\mu_j\| \leq \frac{1}{2}$. Then

  $$\{\sum_j \alpha_j\mu_j : \alpha_j \in \mathbb{C}, \sum_j |\alpha_j| \leq 1\}$$

  is an infinite-dimensional family of Beltrami differentials on U.
- Any Beltrami differential μ on U extends to a unique Γ-invariant Beltrami-differential on $\Gamma.U$ ($\gamma^*\mu = \mu$ for all $\gamma \in \Gamma$) and then to $\overline{\mathbb{C}}$ by taking it to be 0 on the complement of $\Gamma.U$.

The corresponding Kleinian groups

- We start with a Γ-invariant Beltrami differential μ.
- Let φ_μ be the quasi-conformal homeomorphism with $\varphi_\mu^*(0) = \mu$, that is

  $$(\varphi_\mu)_{\overline{z}} = \mu(\varphi_\mu)_z.$$

  Note that this implies $\mu \mapsto \varphi_\mu$ is injective.
- The homeomorphism φ_μ is unique if we normalise it to fix 0, 1 and ∞.
- For any $\gamma \in \Gamma$, $\varphi_\mu \circ \gamma \circ \varphi_\mu^{-1}$ is a Möbius transformation because it is quasi-conformal and

  $$(\varphi_\mu \circ \gamma \circ \varphi_\mu^{-1})^*0 = (\gamma \circ \varphi_\mu^{-1})^*(\mu) = (\varphi_\mu^{-1})^*(\mu) = 0.$$
- So $\varphi_\mu \circ \Gamma \circ \varphi_\mu^{-1}$ is a Kleinian group Γ_μ.

Properties of the map $\mu \mapsto \Gamma_\mu$

- The map $\mu \mapsto \Gamma_\mu$ is holomorphic in μ because $\varphi_\mu \circ \gamma \circ \varphi_\mu^{-1}$ maps 0, ∞ and 1 to $\varphi_\mu(\gamma.0)$, $\varphi_\mu(\gamma.\infty)$ and $\varphi_\mu(\gamma.1)$, and these are holomorphic in μ by the Measurable Riemann Mapping Theorem.
- Since the map $\mu \mapsto \varphi_\mu$ is injective, the map $\mu \mapsto \Gamma_\mu$ is also injective.

  For if $\Gamma_{\mu_1} = \Gamma_{\mu_2}$ and $\varphi_{\mu_1}^{-1} \circ \varphi_{\mu_2} = \varphi$, then $\varphi(\gamma.z) = \gamma.\varphi(z)$ for all $z \in \mathbb{C}$. It follows that φ fixes all fixed points of hyperbolic elements of Γ and must be the identity on L_Γ. Since φ is holomorphic on Ω_Γ, it is holomorphic on $\overline{\mathbb{C}}$ and must be the identity. So $\varphi_{\mu_1} = \varphi_{\mu_2}$ and $\mu_1 = \mu_2$.

Preservation of dimension

- Any holomorphic (or C^1) map from one manifold to another is a submersion onto a submanifold, restricted to any open set on which the rank of the derivative is constant.
- Hence, if μ_λ is any holomorphic family of Beltrami differentials parametrised by an open set Λ of some $\mathbb{C}^n$ then the map $\Phi : \lambda \to \Gamma_{\mu_\lambda}$ is a diffeomorphism restricted to the subset of Λ on which the derivative of Φ has maximal rank.
- Hence
$$\mathrm{dim}\Phi(\Lambda)) \geq \mathrm{dim}(\Lambda).$$
- The dimension of Λ can be taken arbitrarily large and the (complex) dimension of $\Phi(\Lambda)$ is bounded by three times the number of generators of Γ.
- This gives a contradiction, completing the proof that the action of Γ on L_Γ is recurrent. There is no dissipative part.

The recurrent part

The strategy for showing that there is no nontrivial measurable invariant line field on L_Γ is by contradiction. So assume that there is a nontrivial measurable invariant line field on L_Γ.

- By Egoroff's Theorem, there is a compact set K of strictly positive Lebesgue measure restricted to which the line field is continuous.
- By compactness, the line field is uniformly continuous restricted to K. So given $\varepsilon > 0$ there is $\delta > 0$ such that the direction of the line field varies by at most ε on the intersection of K with any ball of radius δ.
- By a basic result in geometric measure theory, almost every point z of K is a Lebesgue density point of K, that is,
$$\lim_{r \to 0} \frac{\mathrm{meas}(K \cap B_r(z))}{\mathrm{meas} B_r(z))} = 1.$$
- Let K_1 be the set of points in K where the density in $B_{r'}(z)$ is at least $1 - \varepsilon_0$ for all $r' \leq r$, choosing r so that K_1 has positive measure.

Continued..

- By recurrence, for a.e. $z \in K_1$, $\gamma.z \in K_1$ for infinitely many γ.
- The aim is to show that the line field cannot vary in direction by $< \varepsilon$ on both $B_\delta(z)$ and $B_\delta(\gamma.z)$.

- Use the compact-abelian-compact decomposition

$$\gamma = \pm P \Delta Q$$

  where

$$\Delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

  with $0 < \lambda < 1$.

- Then for a constant C, either $|\gamma.z - P.0| < C\lambda$ or $|z - Q^{-1}.\infty| < C\lambda$.
- We can assume λ small enough that $2C\lambda < r$.
- In the first case consider the image under γ^{-1} of

$$\{z' : |z' - \gamma.z| < C\lambda\}$$

- If the line field is within ε on proportion $\geq 1 - \varepsilon_0$ of $B_{C\lambda}(\gamma.z)$ then it cannot be so for $B_{C\lambda}(z)$.
- The other case is similar.