Möbius transformations

Möbius transformations are simply the degree one rational maps of $\overline{\mathbb{C}}$:

$$\sigma_A : z \mapsto \frac{az+b}{cz+d} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

where

$$ad - bc \neq 0$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$A \mapsto \sigma_A : GL(2\mathbb{C}) \to \{\text{Mobius transformations }\}$$

is a homomorphism whose kernel is

$$\{\lambda I : \lambda \in \mathbb{C}^*\}.$$

The homomorphism is an isomorphism restricted to $SL(2,\mathbb{C})$, the subgroup of matrices of determinant 1. We have an *action* of $GL(2,\mathbb{C})$ on $\overline{\mathbb{C}}$ by

$$A.(B.z) = (AB).z, \ A, \ B \in GL(2,\mathbb{C}), \ z \in \overline{\mathbb{C}}.$$

The action of $SL(2,\mathbb{R})$ preserves the upper half-plane

$$\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

and also $\mathbb{R} \cup \{\infty\}$ and the lower half-plane. The action of the subgroup

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

preserves the open unit disc, the closed unit disc, and its exterior.

All of these actions are *transitive* that is, for all z and w in the domain there is A in the group with $A.z = w$.

Kleinian groups

Definition 1 A *Kleinian group* is a subgroup Γ of $PSL(2,\mathbb{C})$ which is discrete, that is, there is an open neighbourhood $U \subset PSL(2,\mathbb{C})$ of the identity element I such that

$$U \cap \Gamma = \{I\}.$$

Definition 2 A *Fuchsian group* is a discrete subgroup of $PSL(2,\mathbb{R})$

Equivalently (as usual with topological groups) there is an open neighbourhood V of I such that

$$\gamma V \cap \gamma' V = \emptyset$$

for all γ, $\gamma' \in \Gamma$, $\gamma \neq \gamma'$. To get this, choose V with $V = V^{-1}$ and $V.V \subset U$.

Limit set

Definition 3 The *domain of discontinuity* Ω_Γ of Γ is the set of all $z \in \mathbb{C}$ such that , for some open neighbourhood U of z, $\gamma.U \cap U \neq \emptyset \Leftrightarrow \gamma.z = z$.

Definition 4 The *limit set* L_Γ is the complement of Ω_Γ.

Every Möbius transfomation which is not the identity has 1 or 2 fixed points in $\overline{\mathbb{C}}$

Definition 5 A *hyperbolic* element of Γ, is an element which has two fixed points in $\overline{\mathbb{C}}$ with multipliers at both points off the unit circle. An *elliptic* element has two fixed points with multiplier on the unit circle. A *parabolic* element has just one fixed point.

Properties in brief

- L_Γ is always nonempty, closed and invariant under Γ.
- It is infinite except when Γ is *elementary*, that is, abelian-by-finite. If it is infinite elementary, it can consist of one or two points, depending on whether the infinite order generator is parabolic or hyperbolic.
- If Γ is nonelementary, L_Γ is the closure of the set of fixed points of hyperbolic elements of Γ.
- The domain of discontiuity is open, invariant under Γ and possibly empty.
- Γ acts minimally on L_Γ, that is for every $z \in L_\Gamma$, the set $\{\gamma.z : \gamma \in \Gamma\}$ is dense in L_Γ.
- Γ acts transitively on L_Γ that is, for any open sets U and V intersecting L_Γ, there is $\gamma \in \Gamma$ such that $\gamma.U \cap V \neq \emptyset$.

Extension of the $SL(2,\mathbb{C})$ action

There is an an extension of the $SL(2,\mathbb{C})$ action to upper half space which mimics the action of $SL(2,\mathbb{R})$ on the upper half plane. One neat way of describing the action is to regard upper half space as a subset of the quaternions and to use mutliplication and division in the quaternions. So write

$$H^3 = \{x + yi + tj : t > 0, x, y \in \mathbb{R}\} = \{z + tj : t > 0, z \in \mathbb{C}\}.$$

Then $SL(2,\mathbb{C})$ acts on H^3 by

$$A.w = (aw + b)(cw + d)^{-1} \text{ if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Why does it work?

Note that

$$w^{-1} = \frac{\overline{w}}{|w|^2}$$

where

$$\overline{x + iy + tj + uk} = x - yi - tj - uk,$$

$$|w|^2 = w\overline{w}.$$

Then

$$A.w = \frac{a\overline{c}|w|^2 + b\overline{d} + (ad - bc)w}{|cw + d|^2}$$

Preservation of the hyperbolic metric.

The action of $SL(2, \mathbb{C})$ preserves the metric on H^3 given in classical form by

$$\frac{dx^2 + dy^2 + dt^2}{t^2}.$$

With this metric, H^3 is *hyperbolic space*.The action also preserves the set of hemispheres with centres on the place $\{t = 0\}$ and vertical half-planes — all of which surfaces are *totally geodesic* — and the horizontal planes and spheres in H^3 which are tangent to $\{t = 0\}$.

The action of a Kleinian group on H^3.

The stabiliser of j under the $SL(2, \mathbb{C})$ is the compact group

$$SU(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}.$$

- It follows that if Γ is Kleinian then there is an open neighbourhood U of j in H^3 such that
$$\{\gamma \in \Gamma : \gamma U \cap U \neq \emptyset\} = \{\gamma \in \Gamma : \gamma.j = j\}$$
and this set is finite and consists of finite order elements .
- If Γ has no finite order elements apart from the identity element then this set is simply the identity element.
- Since $SL(2, \mathbb{C})$ acts isometrically, it follows that the action of a Kleinian group on H^3 is discrete.
- If Γ has no finite order elements apart from the identity then H^3/Γ is a *hyperbolic manifold* with covering group Γ.
- If Γ does have finite order elements then H^3/Γ is a *hyperbolic orbifold*

Is there an analogue?

- Analogues of the extension from $\overline{\mathbb{C}}$ to H^3 have been sought in holomorphic dynamics, in particular for rational maps, for example in work of Lyubich and co-workers.
- But there is no easy analogue.
- But we continue the elementary part of the dictionary, promoted by Sullivan in the 1980's.

Action on the domain of discontinuity

We fix a Kleinian group Γ with domain of discontinuity Ω_Γ

- Γ preserves Ω_Γ.
- The stabiliser of a component U of Ω_Γ is a subgroup Γ_1 of Γ.
- Since Γ_1 acts discretely on U, the quotient U/Γ_1 is a Riemann surface, and if Γ_1 has no finite order elements (apart from the identity) then Γ_1 is a quotient group of the covering group.
- If U is simply connected then Γ_1 is the covering group of U/Γ_1.

Here is an analogue of Sullivan's theorem on the nonexistence of wandering domains in the Fatou set for rational functions. the theorem was first proved by Ahlfors (Tulane Symposium on quasiconformal mappings, 1967), but Sullivan gave a proof based on a variaant of his wandering domains proof (Ann of Mathh 122, 1985).

Ahlfors' finiteness theorem *Let Γ be finitely generated. Then for any component U of Ω_Γ with stabiliser Γ_1, U/Γ_1 is always an analytically finite surface, that is, a compact surface minus finitely many punctures. There are only finitely many orbits of the Γ-action in Ω_Γ.*

Analogue of hyperbolicity

We say that Γ acts *hyperbolically* or is *convex cocompact* if one of the two following equivalent properties holds.

- There is a covering of L_Γ by finitely many open balls U_i ($1 \leq i \leq n$) such that, for each $\varepsilon > 0$, there is a covering of L_Γ by sets of the form $\gamma.U_i$ with $\gamma \in \Gamma$ and of radius $< \varepsilon$ in the spherical metric.
- $(H^3 \cup \Omega_\Gamma)/\Gamma$ is compact.

Necessarily, if Γ acts hyperbolically then Γ is finitely generated and every element is either hyperbolic or elliptic.

Cusps

A *maximal parabolic subgroup* in a Kleinian group Γ is the stabiliser Γ_z. of an element $z \in \overline{\mathbb{C}}$, if this group contains at least one parabolic element. If one element in Γ_z is parabolic then all elements are either parabolic or elliptic. The group Γ_z preserves any ball or sphere in H^3 tangent at z. We shall call z a *parabolic point*.

Such balls and spheres are called *horoballs* and *horospheres at z*. There is at least one horoball B at z such that $\gamma.B \cap B \neq \emptyset \Leftrightarrow \gamma \in \Gamma_z$, in which case, of course, $\gamma.B = B$.

The quotient space B/Γ_zin H^3/Γ is called a *cusp neighbourhood*. The following theorem was proved by Sullivan (*Acta Math* 147 1981, 289-299).

Sullivan's finite cusps theorem *Let Γ be finitely generated. There are only finitely many conjugacy classes of maximal parabolic subgroups in Γ.*

Geometrically finite groups

A finitely generated Kleinian group is called *geometrically finite* if for representatives z_i, $1 \leq i \leq n$ of the parabolic point orbits,

$$(H^3 \cup \Omega_\Gamma \cup \Gamma.\{z_i : 1 \leq i \leq n\})/\Gamma$$

is compact. Geometrically finite groups have some nice properties that are reasonably easy to prove.

- Either the limit set is $\overline{\mathbb{C}}$ or it has zero measure.
- If the limit set is connected then it is locally connected.

The first property is now known to hold for all finitely generated Kleinian groups and *not* to hold for rational maps, nor even for polynomials (proved by Buff and Cheritat in 2005).

The second property has been claimed at least for a large class of groups, by Mitra (also known as Brahmachaitanya).

Structural Stability

A convex cocompact group Γ is fairly easily proved to be *structurally stable*, that is, if the generators γ_i of Γ are moved sufficiently little then the resulting group Γ' with generators γ_i' is also Kleinian and quasiconformally conjugate to Γ', that is there is a q-c map $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\varphi(\gamma_i.z) = \gamma_i'.\varphi(z)$$

for all generators γ_i and all $z \in \overline{\mathbb{C}}$.

Structural Stability and no invariant line fields

Sullivan proved the converse. In fact he proved more (Acta Math 155 1985 243-260)

Sullivan's no invariant line field theorem

- *If Γ is finitely generated Kleinian and structurally stable, then every conjugacy to a sufficiently nearby group is quasiconformal.*
- *The quasi-conformal deformation space of any finitely generated Kleinian group is naturally isomorphic to the Teichmuller space of Ω_Γ/Γ.*
- *There are no invariant line fields on the limit set.*

The dictionary on density and structural stablity

- Sullivan was able to prove that all structurally stable Kleinian groups are "good" (convex cocompact) but was unable to prove that structurally stable groups are dense.

- He proved with Mane and Sad that structurally stable rational maps are dense but was unable to prove that structurally stable rational maps are hyperbolic.
- It is now known that geometrically finite groups are dense.(First main results due to Brock and Bromberg.)

  Theorem *Let $M = H^3/\Gamma$ is any hyperbolic 3-manifold such that $\pi_1(M)$ is finitely generated and a representation $\rho : \pi_1(M) \to \Gamma$ is fixed. Then there is a sequence $\rho_n : \pi_1(M) \to \Gamma_n)$ such that $\rho_n \to \rho$ and $H^3/\Gamma_n \to H^3/\Gamma$ and Γ_n is geometrically finite.*
- Density of hyperbolicity in any reasonable family of rational maps is conjectured but still unknown.