Limit set

The definition of limit set of a Kleinian group which is usually given, uses the action on H^3. The definitions given in the previous lecture avoided this.

Definition The *limit set* L_Γ of a Kleinian group Γ is the set of all accumulation points of $\Gamma.w$ for any $w \in H^3$.

Since Γ acts discretely on H^3, the limit set is a closed subset of $\overline{\mathbb{C}}$.

Independence from w.

Lemma 1 *The limit set is independent of the choice of $w \in H^3$.*

Proof The action of Γ preserves the metric

$$\frac{dx^2 + dy^2 + dt^2}{t^2}.$$

Let d denote the corresponding metric on H^3. If w_1, w_2 are two points in H^3 then $d(\gamma.w_1, \gamma.w_2) = d(w_1, w_2)$ for all $\gamma \in \Gamma$. Suppose that $\lim_{n\to\infty} \gamma_n.w_1 = z \in \mathbb{C}$. Writing $\gamma_n.w_1 = z_n + jt_n$ with $z_n \in \mathbb{C}$, we have $z_n \to z$ and $t_n \to 0$. So the Euclidean distance between $\gamma_n.w_1$ and $\gamma_n.w_2 \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \gamma_n.w_1 = \lim_{n\to\infty} \gamma_n.w_2 = z.$$

Compact-abelian-compact decomposition

For $A \in SL(2,\mathbb{C})$, A^*A is has strictly positive eigenvalues with product 1. So we have

$$A^*A = Q^*\Delta^2 Q$$

where $Q \in SU(2,\mathbb{C})$ and

$$\Delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

By choice of Q, we can assume $0 < \lambda < 1$, unless $A \in SU(2,\mathbb{C})$ and A^*A is the identity in which case $\lambda = 1$. Then

$$\sqrt{A^*A} = Q^*\Delta Q,$$

and, for any $\underline{v} \in \mathbb{C}^2$,

$$\|\sqrt{A^*A}\underline{v}\|^2 =< A^*A\underline{v}, \underline{v} >= \|A\underline{v}\|^2.$$

Hence, for some $P \in SU(2,\mathbb{C})$,

$$A = PQ\sqrt{A^*A} = P\Delta Q.$$

This is known as the *compact-abelian-compact decomposition* (of $A \in SL(2,\mathbb{C})$) and is an instance of a decomposition which works for any semisimple Lie group.

Implication for limit set accumulation sequences

Suppose that A_n is a sequence of matrices in a Kleinian group Γ such that $A_n.j$ converges to $z_0 \in \overline{\mathbb{C}}$ and

$$A_n = P_n \Delta_n Q_n$$

with P_n and $Q_n \in SU(2, \mathbb{C})$ and, for $0 < \lambda_n < 1$,

$$\Delta_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix}$$

Then, since $SU(2, \mathbb{C})$ is compact, we can assume after taking a subsequence of $\{A_n\}$, that $P_n \to P \in SU(2, \mathbb{C})$ and $Q_n \to Q \in SU(2, \mathbb{C})$. Then

$$\lim_{n\to\infty} (\Delta_n Q_n).j = 0,$$

and so

$$\lim_{n\to\infty} A_n.j = P.0.$$

Also, if $z \in \overline{\mathbb{C}}$ and $Q.z \neq \infty$ then

$$\lim_{n\to\infty} A_n.z = P.0.$$

Similarly,

$$\lim_{n\to\infty} A_n^{-1}.j = Q^{-1}.\infty,$$

and if $P^{-1}.z \neq 0$,

$$\lim_{n\to\infty} A_n^{-1}.z = Q^{-1}.\infty.$$

We can use this to prove the following:

Lemma 2 *Suppose that Γ is non-elementary. For any $z_0 \in L_\Gamma$,*

$$\overline{\Gamma.z_0} = L_\Gamma.$$

Proof Take any $z \in L_\Gamma$ and sequence $\{A_n\} \subset \Gamma$ with $A_n.j \to z$. Write $A_n = P_n \Delta_n Q_n$ as before and assume $P_n \to P$ and $Q_n \to Q$.

- If $z_0 \neq Q^{-1}.\infty$ then $A_n \to z_0 \to P.0 = z$.
- If $z_0 = Q^{-1}.\infty$ then choose $B \in \Gamma$ such that $B.z_0 \neq z_0$. then $(A_n B).j = A_n.B.j \to z$ by Lemma 1. So $(A_n B).z_0 = A_n.(B.z_0) \to z$.

The complement of the limit set

The following lemma shows that the complement Ω_Γ of L_Γ has a property claimed in the last lecture.

Lemma 3 *If $z \notin L_\Gamma$ then there exists an open neighbourhood U of z such that $A.U \cap U \neq \emptyset$ for $A \in \Gamma$ only if $A.z = z$ – in which case A must be elliptic.*

Proof

- Certainly if z is fixed by A and A is parabolic or hyperbolic then $z \in L_\Gamma$, because if A is parabolic then $A^n.j \to z$ as $n \to \pm\infty$ and if A is hyperbolic then $A^n.j \to z$ either as $n \to +\infty$ or as $n \to -\infty$, depending on whether z is an attracting or repelling fixed point of A.
- So now suppose that there are sequences $\{A_n\} \subset \Gamma$ and $\{z_n\} \subset \overline{\mathbb{C}}$ such that $z_n \to z$ and $A_n.z_n \to z$ and all A_n distinct.
- Write $A_n = P_n \Delta_n Q_n$ as before and as before assume that $P_n \to P$ and $Q_n \to Q$.
- We have seen that $A_n.j \to P.0$ and $A_n^{-1}.j \to Q^{-1}.\infty$, and $A_n.z \to P.0$ unless $z = Q^{-1}.\infty$. But $z \neq Q^{-1}.\infty$ because $z \notin L_\Gamma$.
- In fact $A_n.z' \to P.0$ uniformly on some neighbourhood of z if $z \neq Q^{-1}.\infty$ (which is true). So $A_n.z_n \to P.0$. But then $z = P.0 \in L_\Gamma$, giving a contradiction.

Density of attractive (or repelling) fixed points

The following lemma shows that another property of the limit set claimed in the last lecture is true.

Lemma 4 *If Γ is nonelementary then attractive fixed points of elements of Γ are dense in L_Γ.*

Proof

- Let $z \in L_\Gamma$ and let $\{A_n\} \subset \Gamma$ with $A_n.j \to z$.
- As before write $A_n = P_n \Delta_n Q_n$ with $P_n \to P$ and $Q_n \to Q$, so that $z = P.0$. Choose any $B \in \Gamma$ such that $B.z \neq Q^{-1}.\infty$.
- Then, as in Lemma 2, $A_n B.z' \to z$ uniformly for z' in some open neighbourhood U of z.
- Then for all sufficiently large n, $A_n B(\overline{U}) \subset U$. But then $A_n B$ must be hyperbolic with attractive fixed point in U.
- Since U can be taken arbitrarily small, z is approximated arbitrarily closely by attractive fixed points of hyperbolic elements.

Density of axes

This result has no natural analogue for holomorphic maps but is important in the study of geodesic flows.

Lemma 5 *Let Γ be nonelementary,. Then the set of pairs consisting of attractive and repelling endpoints of hyperbolic elements of Γ is dense in $L_\Gamma \times L_\Gamma$.*

Proof

- Let $(z_1, z_2) \in L_\Gamma \times L_\Gamma$ where z_1 is an attractive fixed point of A_1 and z_2 is a repelling fixed point of A_2 and $z_1 \neq z_2$.

- Let z_3 and z_4 be the repelling and attractive fixed points of A_1 and A_2 respectively. Assume also that $z_4 \neq z_3$.
- Fix disjoint open neighbourhoods U_j of z_j with

$$A_1(\overline{U_1}) \subset U_1, A_1^{-1}(\overline{U_3}) \subset U_3$$

$$A_2^{-1}(\overline{U_2}) \subset U_2,\ A_2(\overline{U_4}) \subset U_4$$

- Then for all sufficiently large n and m,

$$A_2^n.(\overline{U_1}) \subset U_4$$

  and

$$A_1^m(\overline{U_4}) \subset U_1.$$

- So for all sufficiently large n and m

$$A_1^m A_2^n(\overline{U_1}) \subset U_1.$$

  and similarly

$$A_2^{-n} A_1^{-m}(\overline{U_2}) \subset U_2.$$

- It follows that U_1 contains the attractive fixed point of $A_1^m A_2^n$ and U_2 contains the repelling fixed point.

Discussion: Equivalence between hyperbolic action and convex cocompact

- There is a covering of L_Γ by finitely many open balls U_i ($1 \leq i \leq n$) such that, for each $\varepsilon > 0$, there is a covering of L_Γ by sets of the form $\gamma.U_i$ with $\gamma \in \Gamma$ and of radius $< \varepsilon$ in the spherical metric.
- $(H^3 \cup \Omega_\Gamma)/\Gamma$ is compact.