Basic dynamics for Kleinian groups

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Since Γ acts discretely on H^3, the limit set is a closed subset of $\overline{\mathbb{C}}$.

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By choice of Q, we can assume $0 < \lambda < 1$, unless $A \in SU(2,\mathbb{C})$ and A^*A is the identity in which case $\lambda = 1$. Then

$$\sqrt{A^*A} = Q^*\Delta Q,$$

and, for any $\underline{v} \in \mathbb{C}^2$,

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Hence, for some $P \in SU(2, \mathbb{C})$,

$$A = PQ\sqrt{A^*A} = P\Delta Q.$$

This is known as the *compact-abelian-compact decomposition* (of $A \in SL(2, \mathbb{C})$) and is an instance of a decomposition which works for any semisimple Lie group.

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Suppose that A_n is a sequence of matrices in a Kleinian group Γ such that $A_n.j$ converges to $z_0 \in \overline{\mathbb{C}}$ and

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with P_n and $Q_n \in SU(2, \mathbb{C})$ and, for $0 < \lambda_n < 1$,

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Then, since $SU(2, \mathbb{C})$ is compact, we can assume after taking a subsequence of $\{A_n\}$, that $P_n \to P \in SU(2, \mathbb{C})$ and $Q_n \to Q \in SU(2, \mathbb{C})$.

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Also, if $z \in \overline{\mathbb{C}}$ and $Q.z \neq \infty$ then

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- If $z_0 \neq Q^{-1}.\infty$ then $A_n \to z_0 \to P.0 = z$.
- If $z_0 = Q^{-1}.\infty$ then choose $B \in \Gamma$ such that $B.z_0 \neq z_0$. then $(A_nB).j = A_n.B.j \to z$ by Lemma 1. So $(A_nB).z_0 = A_n.(B.z_0) \to z$.

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Lemma 3 *If $z \notin L_\Gamma$ then there exists an open neighbourhood U of z such that $A.U \cap U \neq \emptyset$ for $A \in \Gamma$ only if $A.z = z$ – in which case A must be elliptic.*

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- We have seen that $A_n.j \to P.0$ and $A_n^{-1}.j \to Q^{-1}.\infty$, and $A_n.z \to P.0$ unless $z = Q^{-1}.\infty$. But $z \neq Q^{-1}.\infty$ because $z \notin L_\Gamma$.

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- In fact $A_n.z' \to P.0$ uniformly on some neighbourhood of z if $z \neq Q^{-1}.\infty$ (which is true). So $A_n.z_n \to P.0$. But then $z = P.0 \in L_\Gamma$, giving a contradiction.

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- Then, as in Lemma 2, $A_n B.z' \to z$ uniformly for z' in some open neighbourhood U of z.
- Then for all sufficiently large n, $A_n B(\overline{U}) \subset U$. But then $A_n B$ must be hyperbolic with attractive fixed point in U.

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- Then, as in Lemma 2, $A_n B.z' \to z$ uniformly for z' in some open neighbourhood U of z.
- Then for all sufficiently large n, $A_n B(\overline{U}) \subset U$. But then $A_n B$ must be hyperbolic with attractive fixed point in U.
- Since U can be taken arbitrarily small, z is approximated arbitrarily closely by attractive fixed points of hyperbolic elements.

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- Let z_3 and z_4 be the repelling and attractive fixed points of A_1 and A_2 respectively. Assume also that $z_4 \neq z_3$.

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- It follows that U_1 contains the attractive fixed point of $A_1^m A_2^n$ and U_2 contains the repelling fixed point.

Discussion: Equivalence between hyperbolic action and convex cocompact

- There is a covering of L_Γ by finitely many open balls U_i ($1 \leq i \leq n$) such that, for each $\varepsilon > 0$, there is a covering of L_Γ by sets of the form $\gamma.U_i$ with $\gamma \in \Gamma$ and of radius $< \varepsilon$ in the spherical metric.

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