to our Teacher

On enumeration of meromorphic functions on the line

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Abstract

In 1891 Hurwitz published a conjecture yielding the number of topological types of rational functions on $\mathbf{C}^1$ with fixed orders of poles and fixed critical values assuming the functions Morse on the complement to the poles. Recently there appeared two combinatorial proofs of the conjecture by Goulden and Jackson, and Strehl. We give an independent proof, from the point of view of singularity theory, in the spirit of Arnold's investigations on Laurent polynomials. We are basing on the study of geometry of the moduli space of ordered tuples of points on the line and the properties of the corresponding Lyashko-Looijenga mapping. Also we show that the variety of topological types of Morse functions in our context is an Eilenberg-MacLane $K(\pi, 1)$-space.

The topological type of a meromorphic function on a curve with a non-fixed complex structure is its equivalence class up to homeomorphisms of the domain.

In [7] Hurwitz conjectured a formula (we quote it in section 3.2) giving the number of some special factorisations of a permutation into transpositions.

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This allows one to immediately find the number of topological types of generic meromorphic functions on $\mathbf{C}\mathrm{P}^1$ with fixed orders of poles and fixed critical values.

In [4] the Hurwitz formula, in its particular case of simple poles that is for generic coverings of the Riemann sphere by itself, was rediscovered and proved by Crescimanno and Taylor. Influenced by this work, Goulden and Jackson rediscovered the Hurwitz formula in its full and gave its combinatorial proof without relating the general case to the enumeration of certain functions on the complex line [6]. Shortly after this another combinatorial proof of the formula, which followed Hurwitz's own hints given in [7], was found by Strehl [17].

Unfortunately, it is not easy to trace the natural geometry of the problem behind rather long and complicated calculations of any of the combinatorial proofs, and we failed to do this. On the other hand, the elements of this geometry were uncovered by Arnold in his recent work [1] on Laurent polynomials. There a singularity theory approach considered earlier in [5] was used. Arnold obtained the enumeration of topological types of generic Laurent polynomials with fixed critical values via the study of the mapping which associates to a function the unordered set of its finite critical values.

Such a mapping was first introduced by Lyashko [2, 13] and Looijenga [12] who in particular showed that on the space of monic polynomials of fixed degree it is a finite covering, the number of whose sheets is just the number of the factorisations given by the Hurwitz formula for the cyclic permutations. Since the topological types of generic polynomials are in a one-to-one correspondence with the trees with ordered edges, this also provided an easy proof of the Cayley theorem on enumeration of such trees [12].

In the present paper, we are studying the version of the Lyashko-Looijenga mapping defined on the space of meromorphic functions on $\mathbf{C}\mathrm{P}^1$ with fixed orders of all its n poles. It turns out that it is still a finite covering which is regular out of the bifurcation diagram of functions. This happens in spite of the domain now being non-smooth unlike the case of ordinary or Laurent polynomials: now it contains as a direct factor the base $\mathcal{V}_n$ of the versal deformation of the configuration L_n^n of the coordinate axes in $\mathbf{C}^n$ which is singular in codimension 5 for $n > 3$. To prove the Hurwitz formula basicly means to calculate the degree of the Lyashko-Looijenga mapping. This reduces to the calculation of the (quasi-homogeneous) degree of $\mathcal{V}_n$. Since $\mathcal{V}_n$ projects onto the compactification $\mathcal{W}_n$ of the moduli space of ordered n-tuples of points on the Riemann sphere, the degree in question comes out as the intersection number of certain divisors on $\mathcal{W}_n$.

The approach of the paper looks promising for enumeration of topological types of non-generic meromorphic functions on $\mathbf{C}\mathrm{P}^1$. For this it could be mixed up with the way of consideration of degenerate polynomials introduced in [11, 19, 20]. For a closely related question of the classification of degenerate branched coverings of the 2-sphere up to homeomorphisms of both the source and target see [9].

The paper is organised as follows. In section 1 we reduce the consideration of rational functions on the line with fixed orders of the poles to the singularity problem of studying functions on smoothings of L_n^n. We extract all the information about the base $\mathcal{V}_n$ which is used later on. In section 2 we formulate the theorem on the non-degeneracy of the Lyashko-Looijenga mapping (the proof is postponed until section 4), calculate its degree and state the homotopy type of the complement to the bifurcation diagram of rational functions. In section 3 we relate the obtained degree to the enumeration of the topological types of rational functions, and to the combinatorial problems of enumeration of the graphs with ordered edges and so-called minimal factorisations of a permutation into transpositions. The original Hurwitz formula is that giving the number of such factorisations. Finally, in section 5, we consider an elementary approach to obtain the variety $\mathcal{V}_n$ which avoids a consideration of any versal deformation.

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1 Rational functions and deformations of coordinate axes

1.1 Reduction to smoothings of the coordinate axes arrangement

Any meromorphic function with n poles on $\mathbf{C}\mathrm{P}^1$ can be written in the form

$$p_1(\tfrac{1}{z-\zeta_1}) + p_2(\tfrac{1}{z-\zeta_2}) + \ldots + p_n(\tfrac{1}{z-\zeta_n})\,,$$

where the p_i are polynomials, z is an affine coordinate and the poles are situated at the pairwise distinct finite points ζ_i.

Declare the argument of each of the p_i to be an independent variable z_i. Then the above function is the restriction of the polynomial

$$p_1(z_1) + \ldots + p_n(z_n)$$

to the curve in $\mathbf{C}^n$ given by the quadratic equations

$$z_i z_j + (z_j - z_i)/(\zeta_i - \zeta_j) = 0\,, \quad i \neq j.$$

We include the obtained curve into the family

$$z_i z_j + \varepsilon(z_j - z_i)/(\zeta_i - \zeta_j) = 0\,, \quad i \neq j,$$

and send ε to zero.

This shows that our curve can be considered as a deformation of the curve

$$z_i z_j = 0\,, \quad i \neq j.$$

The latter is the union of n coordinate axes in $\mathbf{C}^n$ usually denoted by L^n_n.

Thus the study of meromorphic functions on $\mathbf{C}\mathrm{P}^1$ with n poles reduces to the study of polynomial functions on smoothings of L^n_n.

1.2 The versal deformation of L^n_n

Consider the miniversal deformation of L^n_n in the form due to D. S. Rim [14, 16]:

$$(z_i - a_{ij})(z_j - a_{ji}) = (a_{i\ell} - a_{ij})(a_{j\ell} - a_{ji})\,, \qquad i,j,\ell > 0,\ i \neq j \neq \ell \neq i, \quad (1)$$

with its base $\mathcal{V}_n$ lying in the coordinate space $\mathbf{C}^{n(n-1)}$ of the a_{ij}, $i \neq j$, and defined there by the conditions

$$(a_{i\ell} - a_{ij})(a_{j\ell} - a_{ji}) = (a_{im} - a_{ij})(a_{jm} - a_{ji}) \tag{2}$$

of independence of the right-hand sides in (1) of the choice of ℓ, and

$$\sum_j a_{ij} = 0\,, \qquad i > 0\,. \tag{3}$$

The base $\mathcal{V}_n$ is $(2n-3)$-dimensional. The $\mathcal{V}_3$ is just $\mathbf{C}^3$. The $\mathcal{V}_4$ is the Segre cone over $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^3$ and thus has an isolated singularity. For $n > 4$, $\mathcal{V}_n$ is singular in codimension 5 and birational to the cone over $\mathbf{C}\mathrm{P}^{n-3} \times \mathbf{C}\mathrm{P}^{n-1}$ [16]. More precisely, the latter concerns the smoothing component of $\mathcal{V}_n$ since, for $n > 5$, it is still a conjecture of Stevens that the mentioned equations for $\mathcal{V}_n$ define a variety without embedded components.

The geometrical sense of the equations of $\mathcal{V}_n$ is as follows. Consider a generic member γ of the versal family. This is a rational curve in $\mathbf{C}^n$ with n punctures. Consider its projection π_i onto the z_i-axis. The punctures project to the points $z_i = a_{ij}$, $j \neq i$, and infinity. The composition $\pi_{i'} \circ \pi_{i''}^{-1}$ obtained from two different projections of γ is a projective transformation of $\mathbf{C}\mathrm{P}^1$ with the n marked points. The quadratic equations (2) of $\mathcal{V}_n$ just mean that the double ratios of the marked points are preserved by the transformation, and vanishing of the sums $\sum_j a_{ij}$ is just a projective normalisation which eliminates possible non-minimality of the deformation.

This also explains why the dimension of $\mathcal{V}_n$ is $2n-3$. Indeed marking of $n-1$ arbitrary finite points with zero sum on, say, the z_1-axis in $\mathbf{C}^n$ provides us with $n-2$ degrees of freedom. To restore the corresponding curve $\gamma \subset \mathbf{C}^n$ from such an ordered set of points, we need to specify $n-1$ projective identifications $\pi_i \circ \pi_1^{-1}$ of the Oz_1 with all the other coordinate axes. Since one of the marked points is sent to the infinity by $\pi_i \circ \pi_1^{-1}$ and the sum on the Oz_i of the others (including that coming from $z_1 = \infty$) has to vanish, there is exactly one degree of freedom in choosing each of the $\pi_i \circ \pi_1^{-1}$.

Remark 1.1 Restriction of any polynomial from $\mathbf{C}^n$ to a smoothing γ of L_n^n can be considered as that of a polynomial $p_1(z_1) + \ldots + p_n(z_n)$. This is a meromorphic function on $\mathbf{C}\mathrm{P}^1$ with n poles at the punctures of γ whose orders are the degrees of the p_i. This reverses the reduction of the previous section.

1.3 The degree of the base

For our further considerations we need some information about the geometry of $\mathcal{V}_n$.

Theorem 1.2 *The degree of the base $\mathcal{V}_n$, $n > 2$, of the versal deformation of L_n^n is n^{n-3}.*

Proof. We have to show that the number in the claim is the number of points in the intersection of $\mathcal{V}_n$ with a generic affine subspace Λ of codimension $2n-3$ in the ambient linear space

$$\mathbf{C}^{n(n-2)} = \{\sum_j a_{ij} = 0, i = 1, \dots, n\} \subset \mathbf{C}^{n(n-1)} .$$

It is convenient to represent the ambient space as a direct product of n spaces

$$\mathbf{C}_i^{n-2} = \{(a_{i1}, \dots, \widehat{a_{ii}}, \dots, a_{in}), \sum_j a_{ij} = 0\} , \qquad i = 1, \dots, n .$$

Projectivising each of these we get a mapping

$$\rho : \mathbf{C}^{n(n-2)} \to \mathcal{P} = \mathbf{C}\mathrm{P}_1^{n-3} \times \dots \times \mathbf{C}\mathrm{P}_n^{n-3} .$$

The equations of $\mathcal{V}_n$ are invariant with respect to individual dilations of the $\mathbf{C}_i^{n-2}$. Hence the image of $\mathcal{V}_n$ under ρ is an $(n-3)$-dimensional variety $\mathcal{W}_n \subset \mathcal{P}$. From the duscussion of the geometrical sense of the equations of $\mathcal{V}_n$ in the previous section, it follows that $\mathcal{W}_n$ is the closure of the space of ordered n-tuples of distinct points on the projective line considered up to automorphisms of the line. Thus $\mathcal{W}_n$ is a compactification of the moduli space of such tuples.

To keep track of the intersection $\mathcal{V}_n \cap \Lambda$ under ρ let us separate the variables in the equations of Λ writing them as

$$\sum_{i=1}^n \alpha_{si} = c_s , \qquad s = 0, \dots, 2n-4 ,$$

where each of the α_{si} is a homogeneous linear form just in $n-1$ variables a_{ij}, and all the constants c_s except for c_0 are just zeros.

Lemma 1.3 *The mapping ρ establishes a one-to-one correspondence between the sets $\mathcal{V}_n \cap \Lambda$ and $\mathcal{W}_n \cap \Lambda'$, where $\Lambda' \subset \mathcal{P}$ is the variety*

$$\operatorname{rank} \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \dots & \dots & \dots \\ \alpha_{2n-4,1} & \dots & \alpha_{2n-4,n} \end{pmatrix} < n \ .$$

Proof of the Lemma. We write a_i for a point of $\mathbf{C}_i^{n-2}$ and $\bar{a}_i$ for its image in $\mathbf{C}\mathrm{P}_i^{n-3}$ under the projectivisation.

For a generic point $(\bar{a}_1, \dots, \bar{a}_n)$ in Λ' (we can assume the set $\mathcal{W}_n \cap \Lambda'$ to contain just such points), the rank condition of the lemma means that there exists a unique point $(b_1 : \dots : b_n) \in \mathbf{C}\mathrm{P}^{n-1}$ such that

$$b_1 \alpha_{s1}(a_1) + \dots + b_n \alpha_{sn}(a_n) = 0 \tag{4}$$

for all $s = 1, \dots, 2n-4$. Thus the line $\{t(b_1 a_1, \dots, b_n a_n), t \in \mathbf{C}\} \subset \mathbf{C}^{n(n-2)}$ satisfies all the homogeneous equations of Λ, and the non-homogeneous one (that with $s = 0$) determines the unique value of t for the unique point of intersection of this line with Λ. $\square$

The points of Λ' are singled out on $\mathcal{W}_n$ as the intersection of $\mathcal{W}_n$ with common zeros of $n-3$ $\quad n \times n$-minors of the matrix in the lemma. Each of the minors is a linear function in a_i when all the other a_j are fixed. Thus each of the $n-3$ divisors corresponding to the minors is the sum $H_1 + \dots + H_n$, where H_i is the lifting to $\mathcal{P}$ of the hyperplane section against the natural projection $\mathcal{P} \to \mathbf{C}\mathrm{P}_i^{n-3}$. Hence the degree we are looking for is just the evaluation of $(H_1 + \dots + H_n)^{n-3}$ on the canonical class $[\mathcal{W}_n]$. Now the claim of the theorem follows from

Lemma 1.4 *For any $(n-3)$-tuple $i_1, \dots, i_{n-3}$ of integers between 1 and n,*

$$(H_{i_1} \cdot \ldots \cdot H_{i_{n-3}})[\mathcal{W}_n] = 1 \ .$$

Proof. Let $\psi_i : \mathcal{W}_n \to \mathbf{C}\mathrm{P}_i^{n-3}$ be the projection onto the ith factor of $\mathcal{P}$. The composition $\psi_{i'} \circ \psi_i^{-1}$ is easily checked to be a Cremona transformation and thus a birational equivalence. Therefore, in general, the intersection of the $n-3$ hyperplanes meets $\mathcal{W}_n$ just at one point and does this transversally. $\square$

Remark 1.5 We conjecture that the space $\mathcal{W}_n$ is in fact the Grothendieck-Knudsen moduli space $\overline{M_{0,n}}$ [10, 8].

1.4 The quasi-homogeneous degree

In what follows we will need a quasi-homogeneous version of Theorem 1.2.

Consider a subvariety V of dimension r in the coordinate space $\mathbf{C}^N$. Assume V to be invariant under the one-parameter group of quasi-homogeneous transformations $(x_1, \ldots, x_N) \mapsto (t^{w_1}x_1, \ldots, t^{w_N}x_N)$, $t \in \mathbf{C} \setminus 0$, where the w_i are positive rational numbers which we call the *weights* of the coordinate functions. Let U be the set of common solutions of r equations

$$f_i(x) = c_i\,, \quad i = 1, \ldots, r\,,$$

where the f_i are generic quasi-homogeneous functions of degrees d_i and the c_i constants. Assume that U is a complete intersection which meets V just transversally, and $\mu = \mu(d_1, \ldots, d_r)$ is the number of points in $V \cap U$. Then the ratio

$$\mu / \prod_{i=1}^{r} d_i$$

does not depend on the choice of the d_i. This number will be called the *quasi-homogeneous degree of V (with respect to the weights w_j).*

Example 1.6 The quasi-homogeneous degree of $\mathbf{C}^N$ itself is $1/\prod_{j=1}^{N} w_j$.

Now we return to our variety $\mathcal{V}_n$. Take arbitrary natural numbers $k_1, \ldots,$ k_n, and assign weights $1/k_i$ to the coordinates a_{ij} on $\mathbf{C}^{n(n-1)} \supset \mathbf{C}^{n(n-2)} \supset \mathcal{V}_n$.

Theorem 1.7 *The quasi-homogeneous degree of $\mathcal{V}_n$, $n > 2$, with respect to the chosen weights of the coordinate functions is*

$$(k_1 + \ldots + k_n)^{n-3} \cdot k_1 \cdot \ldots \cdot k_n\ .$$

Proof. This will be almost word-to-word to that of Theorem 1.2. We denote similar objects by the same symbols and just point out the arising differences.

This time for the variety Λ we take the set of common solutions of the Pham-type equations

$$\alpha_{s1} + \ldots + \alpha_{sn} = c_s\,, \quad s = 0, \ldots, 2n-4,$$

where each α_{si} is a generic homogeneous degree k_i form in the coordinates on $\mathbf{C}_i^{n-2}$ which does not depend on the other variables, and, as earlier, all the constants except for c_0 vanish.

We have to show that the number of points in $\mathcal{V}_n \cap \Lambda$ is that appearing in our present theorem.

The first correction to adjust the proof of Theorem 1.2 for the quasi-homogeneous case arises in the analog of Lemma 1.3. This time the projection ρ provides a mapping of degree $k_1 \cdot \ldots \cdot k_n$ from $\mathcal{V}_n \cap \Lambda$ to $\mathcal{W}_n \cap \Lambda'$. Indeed, equations (4) now imply

$$\alpha_{s1}((tb_1)^{1/k_1} a_1) + \ldots + \alpha_{sn}((tb_n)^{1/k_n} a_n) = 0\,, \quad s > 0,$$

for some $t \in \mathbf{C}$ which is uniquely determined by the 0th equation:

$$\alpha_{01}((tb_1)^{1/k_1} a_1) + \ldots + \alpha_{0n}((tb_n)^{1/k_n} a_n) = c_0$$

$$\Longleftrightarrow \quad tb_1\alpha_{01}(a_1) + \ldots + tb_n\alpha_{0n}(a_n) = c_0\,.$$

The second correction to be done is that the $n-3$ $\quad n \times n$-minors defining the meeting points of Λ' and $\mathcal{W}_n$ are now each of the class $k_1H_1 + \ldots + k_nH_n$. $\square$

2 Lyashko-Looijenga mapping

2.1 Bifurcation diagram of rational functions

From the above discussion, we see that a rational function on CP^1 with $n > 1$ poles of fixed orders $k_1, \ldots, k_n$ is a polynomial

$$\begin{aligned} P_\lambda(z_1, \ldots, z_n) = z_1^{k_1} + \lambda_{1,1} z_1^{k_1-1} + \ldots + \lambda_{1,k_1-1} z_1 + \ldots + \\ + z_n^{k_n} + \lambda_{n,1} z_n^{k_n-1} + \ldots + \lambda_{n,k_n-1} z_n + \lambda_0 \end{aligned} \tag{5}$$

restricted to a smoothing of L_n^n (we reduce the coefficients of the $z_i^{k_i}$ to 1s by rescaling of the z_i).

Closing the space of such restrictions by allowing non-smooth curves we arrive at consideration of the family of functions on curves in $\mathbf{C}^n$ over the base $\mathcal{V}_n \times \mathbf{C}_\lambda^{k-n+1}$ (from now on we set $k = k_1 + \ldots + k_n$). This family will be the main object of our study.

A generic member of the family is a degree k Morse function on a rational curve with n punctures. Due to the Riemann-Hurwitz formula, it has $k+n-2$ distinct critical values. Note that the dimension of the base $\mathcal{V}_n \times \mathbf{C}_\lambda^{k-n+1}$ is also $k+n-2$, the fact which is heavily exploited in what follows.

Definition 2.1 *The set* $\Sigma \subset \mathcal{V}_n \times \mathbf{C}_\lambda^{k-n+1}$ *of functions having less than* $k+n-2$ *distinct critical values is called the* bifurcation diagram of rational functions.

There are three generic ways to drop the number of critical values corresponding to the three components of the hypersurface Σ:

Σ_m, *Maxwell stratum*: two different critical points on a smooth curve are on the same level;

Σ_c, *caustic*: there is a degenerate critical point on a smooth curve;

Σ_s: the curve is not smooth.

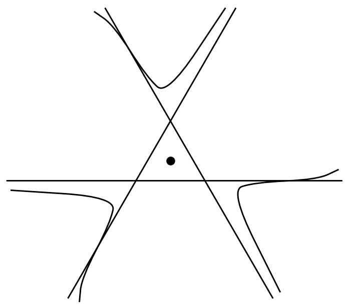
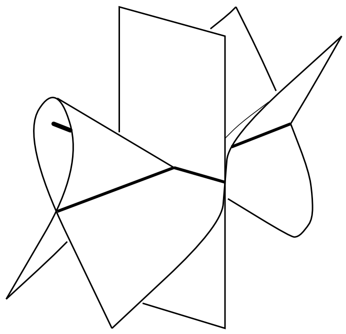
Note that the pair $(\mathcal{V}_n \times \mathbf{C}_\lambda^{k-n+1}, \Sigma)$ is trivial in the λ_0-direction.

Example 2.2 The family of rational functions with 3 simple poles consists of the restrictions of the functions $x+y+z+\delta$ from $\mathbf{C}^3$ to the family of curves defined by the condition

$$\operatorname{rank}\begin{pmatrix} x & y & \alpha \\ \beta & y+\gamma & z \end{pmatrix} < 2\,.$$

Here the base is smooth and $\alpha, \beta, \gamma, \delta$ are the parameters. Due to the mentioned triviality the bifurcation diagram is well-represented by its intersection with the 3-space $\delta = 0$.

The projectivisation in $\mathbf{C}\mathrm{P}^2 = \{(\alpha : \beta : \gamma : 0)\}$ of the bifurcation diagram is a nodal cubic Σ_c with the tangent lines at all its three inflection points forming the component Σ_s. This is shown on the left-hand side of the figure below (the node of the cubic is the isolated real point in the centre). On the right-hand side of the figure we give the spatial version of the other real form of the diagram.

2.2 The covering

The Viéte mapping identifies the set of all unordered r-tuples of complex numbers with the space $\mathbf{C}^r$ of monic polynomials in one variable of degree r. This space contains the *discriminant* hypersurface Δ of polynomials with multiple roots.

Definition 2.3 The Lyashko-Looijenga mapping *for rational functions is the mapping*

$$\mathcal{L}: \ \mathcal{V}_n \times \mathbf{C}^{k-n+1} \ \to \ \mathbf{C}^{k+n-2}$$

which sends a function on a curve to the unordered set of its critical values counted with the multiplicities.

Of course, at first this mapping is defined just as a mapping

$$\mathcal{L}': \quad (\mathcal{V}_n \times \mathbf{C}^{k-n+1}) \setminus \Sigma \quad \to \quad \mathbf{C}^{k+n-2} \setminus \Delta$$

which sends a Morse function on a smooth curve to an unordered $(k+n-2)$-tuple of distinct numbers. Then one can easily verify (see, e.g. [5, 1]) that $\mathcal{L}'$ extends to generic points of Σ. For example, the value of a generic function at a node has to be counted as critical of multiplicity 2. In what follows we are using both $\mathcal{L}$ and $\mathcal{L}'$.

Theorem 2.4 *The Lyashko-Looijenga mapping $\mathcal{L}$ is a proper finite covering. As a mapping from $(\mathcal{V}_n \times \mathbf{C}^{k-n+1}) \setminus \Sigma$ to $\mathbf{C}^{k+n-2} \setminus \Delta$ it has no branching.*

The version of this theorem for ordinary polynomials in one variable is the theorem of Lyashko and Looijenga [3, 12]. For Laurent polynomials it was proven in [5]. We prove Theorem 2.4 for arbitrary $n > 2$ in Section 4. Now we extract some corollaries from it.

Corollary 2.5 *The complement* $(\mathcal{V}_n \times \mathbf{C}^{k-n+1}) \setminus \Sigma$ *to the bifurcation diagram of rational functions is an Eilenberg-MacLane* $K(\pi, 1)$*-space, where* π *is a subgroup of index*

$$k^{n-3}(k+n-2)! \prod_{i=1}^{n} \frac{k_i^{k_i}}{(k_i-1)!}$$

in the Artin's group $B(k+n-2)$ *of braids on* $k+n-2$ *threads.*

Proof. We have to show that the number in the corollary is the degree of the Lyashko-Looijenga mapping.

The mapping $\mathcal{L}$ is quasi-homogenious: its coordinate functions get weights $1, 2, \ldots, k+n-2$ when we assign the following weights to the parameters involved (recall the settings of (1) and (5)):

$$\mathrm{wt}\, a_{ij} = 1/k_i\,, \qquad \mathrm{wt}\, \lambda_{is} = s/k_i\,, \qquad \mathrm{wt}\, \lambda_0 = 1\,.$$

Since the quasi-homogenious degree of a direct product of two varieties is the product of their quasi-homogenious degrees, applying Theorem 1.7 and Example 1.6 we have

$$\begin{aligned} \deg \mathcal{L} &= (k+n-2)! \cdot \deg\,(\mathcal{V}_n \times \mathbf{C}^{k-n+1}) \\ &= (k+n-2)! \cdot \deg \mathcal{V}_n \cdot \deg \mathbf{C}^{k-n+1} \\ &= (k+n-2)! \cdot k^{n-3} \prod_{i=1}^{n} k_i / \prod_{i=1}^{n} \frac{(k_i-1)!}{k_i^{k_i-1}} \end{aligned}$$

which is the required number. □

Remark 2.6 The corollary remains valid for both ordinary, with $k > 1$, and Laurent polynomials [3, 12, 5, 1] in spite of those cases not being covered by Theorem 1.7.

Remark 2.7 The family of functions (5) on the family of curves (1–3) which we studied in this section is a miniversal deformation of the function $z_1^{k_1} + \ldots + z_n^{k_n}$ on the arrangement of the coordinate axes in $\mathbf{C}^n$ for the natural equivalence of functions on curves. The setting here involves deforming both a function and a curve. This equivalence will be a subject of a separate paper.

3 Topological types of rational functions

3.1 The enumeration

Consider two holomorphic mappings, f and f', from closed complex curves Γ and Γ' to $\mathbf{C}\mathrm{P}^1$. We say that they are of *the same topological type* if there exists a commutative diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma' \\ f \searrow & & \swarrow f' \\ & \mathbf{C}\mathrm{P}^1 & \end{array}$$

in which the horizontal arrow is a homeomorphism.

Mark a point (infinity) on the $\mathbf{C}\mathrm{P}^1$ and call its inverse images *poles.*

Theorem 3.1 *The number of topological types of meromorhpic functions on $\mathbf{C}\mathrm{P}^1$ with poles of orders $k_1, \ldots, k_n$ and fixed critical values, assuming that on each finite critical level there is just one critical point and this point is Morse, is*

$$M(k_1, \ldots, k_n) = \frac{k^{n-3}\,(k+n-2)!}{N} \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \ .$$

Here $k = k_1 + \ldots + k_n$ and $N = n_1! \cdot \ldots \cdot n_s!$, where the n_j are the cardinalities of the sets of equal numbers among the k_i, $n_1 + \ldots + n_s = n$.

Remark 3.2 To be precise there are three cases not covered by the theorem: $M(1) = M(2) = M(1,1) = 1$.

Proof. We assume $n > 2$. For the cases $n = 1, 2$ see [1].

All the topological types of Theorem 3.1 are present in the fibre of the Lyashko-Looijenga mapping $\mathcal{L}'$. The points in the fibre giving the same type are obtained from each other by permuting the coordinates z_j in $\mathbf{C}^n$ (see (5)) corresponding to the poles of the same order, and by multiplying z_i by a root of unity of order k_i. This provides the action of the group

$$G = S_{n_1} \times \ldots \times S_{n_r} \times \mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n}$$

on $\mathcal{V}_n \times \mathbf{C}^{k-n+1} \setminus \Sigma$ (here S_q is the full symmetric group on q elements). The action is easily verified to be free. The Lyashko-Looijenga mapping $\mathcal{L}'$ factors through the factorisation π_G by this action:

$$\mathcal{V}_n \times \mathbf{C}^{k-n+1} \setminus \Sigma \quad \xrightarrow{\pi_G} \quad \mathcal{T}_{k_1,\ldots,k_n} \quad \xrightarrow{L} \quad \mathbf{C}^{k+n-2} \setminus \Delta \ ,$$

$$\mathcal{L}' = L \circ \pi_G \ .$$

Here $\mathcal{T}_{k_1,\ldots,k_n}$ is the variety of all the topological types when the distinct Morse critical levels vary. The number of types under the question is the degree of the covering L which is the ratio of the degree of $\mathcal{L}'$ and order of G. Now the claim follows from Corollary 2.5. $\square$

As a by-product we have obtained:

Corollary 3.3 (cf. [1]) *The variety $\mathcal{T}_{k_1,\ldots,k_n}$ of all topological types of meromorphic functions on $\mathbf{C}\mathrm{P}^1$ with poles of orders $k_1, \ldots, k_n$, $k = k_1 + \ldots + k_n$, and with $k + n - 2$ distinct critical values is naturally covered by the space $\mathcal{V}_n \times \mathbf{C}^{k-n+1} \setminus \Sigma$ of Morse functions on smoothings of L_n^n with the poles of the same orders. The degree of this unramified covering is*

$$n_1! \cdot \ldots \cdot n_s! \cdot \prod_{i=1}^{n} k_i \ .$$

Moreover, $\mathcal{T}_{k_1,\ldots,k_n}$ is itself the space of an unramified covering of $\mathbf{C}^{k+n-2} \setminus \Delta$ of degree $M(k_1, \ldots, k_n)$.

Corollary 3.4 *The variety $\mathcal{T}_{k_1,\ldots,k_n}$ of the topological types is an Eilenberg-MacLane $K(\pi, 1)$-space for a subgroup π of index $M(k_1, \ldots, k_n)$ in the braid group $B(k + n - 2)$.*

3.2 Graphs associated to meromorphic functions

A way to associate a graph to a meromorphic function on a complex curve was suggested first by Zdravkovska in [18]. This approach allows one to establish an equivalence between the topological types of meromorphic functions and classification of certain graphs. Our exposition below follows that of Arnold [1].

Consider a holomorphic degree k mapping $f : \Gamma \to \mathbf{C}\mathrm{P}^1$ of a connected compact Riemann curve. Assume it is a Morse function on the complement to the poles. Take an arbitrary finite non-critical value t_* of f. Its inverse images will be the k vertices of our graph. The number of the edges of the graph is going to be equal to the number m of finite critical values of f.

The naturally ordered edges come out when we connect t_* with all the finite critical values $t_1, \dots, t_m$ of f by a system of m paths t_*t_i in $\mathbf{C}$ without mutual and self-intersections. Here, after choosing the 1st critical value, the indices are assigned to the others according to the counter-clockwise order in which the paths leave t_*. Now the edge number i connects the two points of $f^{-1}(t_*)$ merging in the homotopy of the fibres $f^{-1}(t)$ while t follows the path t_*t_i.

The graph thus obtained is connected since Γ was such. It follows immediately from the Riemann theorem that any connected graph with ordered edges can be obtained in this way. By the Riemann-Hurwitz theorem, the genus g of the curve Γ involved is determined by the formula

$$2 - 2g = k - m + n\,,$$

where n is the number of poles of f.

The monodromy interpretation relates to the ith edge of the graph the transposition τ_i of its endpoints. The permutation $\sigma = \tau_m \circ \dots \circ \tau_1 \in S_k$ is called the *Coxeter element* of the graph [1]. If $k_1, \dots, k_n$ are the orders of the poles of f, $k_1 + \dots + k_n = k$, then σ has cyclic type $(k_1, \dots, k_n)$, that is it is the product of n independent cycles of these orders.

Theorem 3.5 ([1]) *Consider meromorphic functions on connected Riemann curves which have n poles of orders $k_1, \dots, k_n$, are Morse on the complement to the poles and have m fixed finite critical values. The number of topological types of such functions is equal to the number of graphs with $k_1+\dots+k_n$ vertices and m ordered edges whose Coxeter element has cyclic type $(k_1, \dots, k_n)$.*

Thus the number $M(k_1, \dots, k_n)$ in Theorem 3.1 is the number of graphs with k vertices and $k + n - 2$ ordered edges whose Coxeter element has cyclic type $(k_1, \dots, k_n)$, $k_1 + \dots + k_n = k$. In fact, at this point the history developed in the opposite direction: for the first time this number came out as the number of the topological types within the interpretation given in [15] to the combinatorial result of Goulden and Jackson [6] related to the enumeration of graphs with ordered edges.

The setting of [6], repeating that by Hurwitz [7], was as follows.

Consider a permutation σ of cyclic type $(k_1, \dots, k_n)$ on the set $\{1, 2, \dots, k\}$, $k = k_1 + \dots + k_n$. Denote by $\widehat{M}(\sigma)$ the number of its factorisations into a product of transpositions $\sigma = \tau_m \circ \dots \circ \tau_1$ such that

- $\tau_1, \dots, \tau_m$ generate the symmetric group S_k;
- m is minimal with respect to the previous requirement.

The first requirement means that σ is the Coxeter element $C(\Theta)$ of some connected edge-ordered graph Θ (equipped with an appropriate order of its vertices so that $\sigma = C(\Theta)$). The second requirement corresponds to the case of the genus zero curve.

Theorem 3.6 ([6], Hurwitz conjecture [7])

$$\widehat{M}(\sigma) = k^{n-3}(k+n-2)! \prod_{i=1}^{n} \frac{k_i^{k_i}}{(k_i-1)!}\,.$$

Proof. To obtain $\widehat{M}(\sigma)$ from the number $M(k_1, \dots, k_n)$ of the graphs with ordered edges is to count the ambiguity in ordering the vertices of the graph Θ for which $C(\Theta) = \sigma$:

- having a cycle of length k_i in σ we can assign its elements in the proper order to the vertices of Θ participating in any cycle of $C(\Theta)$ of the same length;
- this proper order is defined just up to a cyclic permutation. $\square$

The given argument just repeats the factorisation π_G of the previous section.

Remark 3.7 Note that $\widehat{M}(\sigma)$ is the number of Corolary 2.5, that is the degree of the Lyashko-Looijenga mapping $\mathcal{L}$.

4 Proof of Theorem 2.4

To prove Theorem 2.4 we need to show that $\mathcal{L}^{-1}(0) = 0$ and that the mapping $\mathcal{L}$ is a local diffeomorphism out of the bifurcation diagram.

We will assume $n > 2$. For $n = 1, 2$ see [3, 12, 5, 1].

4.1 The inverse image of the origin

We are searching for points in the base $\mathcal{V}_n \times \mathbf{C}^{k-n+2}$ of our family (Section 2.1) corresponding to the pairs {curve γ, function P_λ on it} in which the function has just one critical value, zero. Its total multiplicity is $k+n-2$.

First assume γ being smooth. Expressing all the variables in terms of, say, z_1 we write function P_λ as a ratio of polynomials $f(z_1)/g(z_1)$ where f is of degree k. For this ratio to have critical value 0 of multiplicity r at some point, polynomial f must have a root of multiplicity $r+1$ at the same point. Thus the number of distinct critical points is $2-n<0$.

Now let curve γ be singular. This means that it consists of smoothings γ_r of curves $L_{n_r}^{n_r}$, $n_r>0$, $\sum_r n_r = n$, each lying in its own affine n_r-dimensional complex subspace $\mathbf{C}^{n_r}$ whose direct sum is the ambient $\mathbf{C}^n$. The $\mathbf{C}^{n_r}$ are just parallel translations of the corresponding coordinate planes in $\mathbf{C}^n$. The points of pairwise intersections of the $\mathbf{C}^{n_r}$ are exactly the points where the γ_r meet each other providing the singularities of γ. The values of P_λ at these points are considered as critical. The curve γ is connected.

Let γ' be one of the smooth curves γ_r. Consider the restriction P' of P_λ to γ'. In fact this is the restriction of the polynomial from the corresponding $\mathbf{C}^{n'} = \mathbf{C}^{n_r}$ of the same sort as P_λ itself (see (5)) but just in n' variables. Similarly to the setting of k, let k' be the sum of the degrees of the summands of P'.

As earlier, the sum of the multiplicities of all the critical values of P' is $k'+n'-2$, and we want all these values to be zero. Let d be the number of singular points of γ situated on γ' which are not critical for P'. We want P' to vanish at them as well. Besides this, function P' can have some other simple zeros.

Considering P' as a ratio of polynomials in just one variable, with the numerator of dergee k', we see that our requirements imply that the number of critical points of P' is at most $2-n'-d$. Thus $n'=1$, function P' being a polynomial on $\gamma' = \mathbf{C}^1$, and only one singular point of γ is on γ'. This point is the only zero of P'.

Since γ' was an arbitrary component of γ, this forces $\gamma = L_n^n$ and $\lambda = 0$. Hence $\mathcal{L}^{-1}(0) = 0$.

4.2 Local diffeomorphism

We will show that $\mathcal{L}$ is a local diffeomorphism out of the bifurcation diagram calculating a related Jacobian matrix in some coordinate representation and showing that it cannot degenerate.

Let us slightly alter the realisation of the variety $\mathcal{V}_n$ in the space $\mathbf{C}^{n(n-1)}$, with the coordinates a_{ij}, $i \neq j$, taking for it the same quadratic equations (2) as in Section 1.2, but changing the n linear ones to

$$a_{1n} = a_{2n} = \ldots = a_{n-1,n} = a_{n1} = 0 \ .$$

Now we choose a local chart on $\mathcal{V}_n$ so that any smoothing of L_n^n would show up in this chart. For such a chart the $2n-3$ coordinates $a_{12}, a_{13}, \ldots, a_{1,n-1}, a_{21}, a_{31}, \ldots, a_{n-1,1}, a_{n,n-1}$ can be taken.

Our settings sum up to the following data in the pole matrix:

$$(a_{ij}) = \begin{pmatrix} \infty & * & * & \ldots & * & * & 0 \\ * & \infty & ? & \ldots & ? & ? & 0 \\ * & ? & \infty & \ldots & ? & ? & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ * & ? & ? & \ldots & \infty & ? & 0 \\ * & ? & ? & \ldots & ? & \infty & 0 \\ 0 & ? & ? & \ldots & ? & * & \infty \end{pmatrix} .$$

Here the stars mark the coordinates in the chosen chart. Given particular values of the marked elements (so that there are no given coinciding numbers in any of the lines) all the other entries of the matrix (question marks) can be restored from the quadratic equations of $\mathcal{V}_n$.

We start our calculations. Solving the relations

$$(z_i - a_{ij})(z_j - a_{ji}) = (a_{i\ell} - a_{ij})(a_{j\ell} - a_{ji}) \, ,$$

with $i = 1$ and appropriate ℓ, we express all the z_j in terms of z_1:

$$z_j = a_{j1} + \frac{(a_{1n} - a_{1j})(a_{jn} - a_{j1})}{z_1 - a_{1j}} = \frac{a_{j1} z_1}{z_1 - a_{1j}} \ , \quad j = 2, \ldots, n-1,$$

and

$$z_n = a_{n1} + \frac{(a_{1,n-1} - a_{1n})(a_{n,n-1} - a_{n1})}{z_1 - a_{1n}} = \frac{a_{1,n-1} a_{n,n-1}}{z_1} \, .$$

This transfers the family $P_\lambda(z_1, \ldots, z_n)$ of (5) into a family of functions $Q_{\lambda,a}(z)$ in just one variable $z = z_1$ with the parameters $\lambda \in \mathbf{C}^{k-n+1}$ and the above chosen $2n-3$ of the a_{ij}. The mapping $\mathcal{L}$ being a local diffeomorphism is equivalent to the matrix of the velocities of all the $k+n-2$ critical values of $Q_{\lambda,a}$ being non-degenerate once the critical values are distinct.

The latter velocities are just the first derivatives of Q with respect to all its $k+n-2$ parameters evaluated at its $k+n-2$ critical points. Writing $Q_{\lambda,a}(z)$ as $p_1(z_1(z)) + \ldots + p_n(z_n(z)) + \lambda_0$ (cf. (5)), we see that those derivatives are

$$z^{k_1-1}, z^{k_1-2}, \ldots z, z_2^{k_2-1}, z_2^{k_2-2}, \ldots z_2, \ldots, z_n^{k_n-1}, z_n^{k_n-2}, \ldots z_n, 1,$$

and

$$\frac{\partial Q}{\partial a_{j1}} = \frac{z_j}{a_{j1}} \frac{\partial p_j}{\partial z_j}(z_j), \quad 2 \le j \le n-1,$$

$$\frac{\partial Q}{\partial a_{1j}} = \frac{z_j}{z - a_{1j}} \frac{\partial p_j}{\partial z_j}(z_j), \quad 2 \le j \le n-2,$$

$$\frac{\partial Q}{\partial a_{1,n-1}} = \frac{z_{n-1}}{a_{1,n-1}} \frac{\partial p_{n-1}}{\partial z_{n-1}}(z_{n-1}) + \frac{a_{n,n-1}}{z} \frac{\partial p_n}{\partial z_n}(z_n),$$

$$\frac{\partial Q}{\partial a_{n,n-1}} = \frac{a_{1,n-1}}{z} \frac{\partial p_n}{\partial z_n}(z_n).$$

Up to addition of the other velocities and multiplication by non-zero constants, $\frac{\partial Q}{\partial a_{j1}}$ can be replaced by $z_j^{k_j}$, $\frac{\partial Q}{\partial a_{1j}}$ by $z_j^{k_j+1}$, $2 \le j \le n-1$ (here one needs to recall what $z_j(z)$ is), and $\frac{\partial Q}{\partial a_{n,n-1}}$ by $z_n^{k_n}$.

Doing similar linear transformations of the velocities once again we arrive at the set of $k+n-2$ functions

$$\begin{gathered} z^{k_1-1}, z^{k_1-2}, \ldots z, 1, z^{-1}, \ldots, z^{-k_n+1}, z^{-k_n}, \\ (z-a_{12})^{-(k_2+1)}, (z-a_{12})^{-k_2}, \ldots, (z-a_{12})^{-1}, \ldots, \\ (z-a_{1,n-1})^{-(k_{n-1}+1)}, (z-a_{1,n-1})^{-k_{n-1}}, \ldots, (z-a_{1,n-1})^{-1}. \end{gathered} \tag{6}$$

For the diffeomorphness required it is sufficient to show that those functions are linearly independent at arbitrary $k+n-2$ distinct points on the z-line once they are defined at those points. Note that all the a_{1j} are distinct and non-zero.

Multiplying all the functions in (6) by their common denominator we get $k+n-2$ polynomials of degree less than $k+n-2$ which are linearly independent. Indeed any solution $g=g(z)$ of the differential equation

$$\frac{d^{k+n-2}g}{dz^{k+n-2}}=0$$

with prescribed values of its derivatives of orders $0,1,\ldots,k_1+k_n-1$ at $z=0$, and of orders $0,1,\ldots,k_j$ at $z=a_{1j}$, $j=2,\ldots,n-1$, is a linear combination of these polynomials. Now the matrix of values at r distinct points of any r functions forming a linear basis in the space of polynomials in one variable of degree less than r is non-degenerate.

This shows that the critical locus of the Lyashko-Looijenga mapping is contained in the bifurcation diagram Σ of rational functions. In fact it is the union of the caustic and the Maxwell stratum [5, 1].

5 An elementary approach to the variety $\mathcal{V}_n$

Here we show how, in the study of meromorphic functions on the projective line, to come to the consideration of the base of the miniversal deformation of the arrangement of the coordinate axes in $\mathbf{C}^n$ without using any notion of a versal deformation at all.

First of all note that a function on $\mathbf{C}\mathrm{P}^1$ with a simple pole at a point ξ is an affine coordinate z on the line, $z(\xi)=\infty$. Hence a meromorphic function with distinct poles $\xi_1,\ldots,\xi_n\in\mathbf{C}\mathrm{P}^1$ is a function

$$p_1(z_1)+\ldots+p_n(z_n)+\lambda_0\,,$$

where the z_i are (dependent) affine coordinates, $z_i(\xi_i)=\infty$, $\lambda_0\in\mathbf{C}$, and the polynomials p_i may be assumed to have no free terms and their degrees are the orders of the poles.

We set

$$a_{ij}=z_i(\xi_j)\,,\qquad j\neq i\,,$$

being the coordinates of the finite poles in the charts.

For $n>1$, we get rid of the most of the ambiguity in our choice of the z_i fixing the origins by the requirements

$$\sum_{j,j\neq i}a_{ij}=0\,,$$

and leaving just a finite number of options after rescaling the z_i to make the highest coefficient of each of the p_i to be 1 (cf. (5)).

Since the coordinate change $z_i = z_i(z_j)$ is an automorphism of the projective line, the coordinates z_i of a point of $\mathbf{C}\mathrm{P}^1$ in different charts are subject to the doulbe-ratio relations in which $a_{ii} = a_{jj} = \infty$:

$$\frac{z_i - a_{ij}}{a_{i\ell} - a_{ij}} \cdot \frac{a_{i\ell} - a_{ii}}{z_i - a_{ii}} = \frac{z_j - a_{jj}}{a_{j\ell} - a_{jj}} \cdot \frac{a_{j\ell} - a_{ji}}{z_j - a_{ji}},$$

that is

$$(z_i - a_{ij})(z_j - a_{ji}) = (a_{i\ell} - a_{ij})(a_{j\ell} - a_{ji}), \tag{7}$$

In particular, this holds for the coordinates of the pole ξ_m:

$$(a_{im} - a_{ij})(a_{jm} - a_{ji}) = (a_{i\ell} - a_{ij})(a_{j\ell} - a_{ji}).$$

Thus we are ending up with the quadratic equations which, along with the above conditions of vanishing of the sums $\sum_j a_{ij}$, are just the equations (3) and (2) of the variety $\mathcal{V}_n$. The deformation (7) is exactly what we called the miniversal deformation of L_n^n in section 1.2.

Allowing the ξ_i to vary still being distinct and taking into account that the a_{ij} are well-defined only modulo the multiplication by the roots of unity of the order equal to the degree of p_i, we see that the closure of the set of points $(a_{12}, \ldots, a_{n,n-1}) \in \mathbf{C}^{n(n-1)}$ which we are able to obtain from the rational functions in the above way is just the smoothing component of $\mathcal{V}_n$.

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