Working Paper in Economics

201910

October 2019

Innovation and Inequality from Stagnation to Growth

Angus C. Chu
Pietro Peretto

Innovation and Inequality from Stagnation to Growth

Angus C. Chu, University of Liverpool
Pietro Peretto, Duke University

September 2019

Abstract

This study explores the evolution of income inequality in an economy featuring an endogenous transition from stagnation to growth. We incorporate heterogenous households into a Schumpeterian model of endogenous takeoff. In the pre-industrial era, the economy is in stagnation, and income inequality is determined by an unequal distribution of land ownership and remains stationary. When takeoff occurs, the economy experiences innovation and economic growth. In this industrial era, income inequality gradually rises until the economy reaches the balanced growth path. Finally, we calibrate the model for a quantitative analysis and compare the simulation results to historical data in the UK.

JEL classification: D30, O30, O40
Keywords: income inequality, innovation, economic growth, endogenous takeoff

Chu: anguscccc@gmail.com. Management School, University of Liverpool, Liverpool, United Kingdom. Peretto: peretto@econ.duke.edu. Department of Economics, Duke University, Durham, United States. We are extremely grateful to Jakob Madsen for sharing his data with us and Xilin Wang for his excellent research assistance.

1 Introduction

In his seminal work, Kuznets (1955) hypothesizes that industrialization increases income inequality. Analyzing historical data in Britain, Williamson (1980, 1985) provides evidence for this hypothesis that income inequality increases after the Industrial Revolution and keeps rising until the mid-19th century.[1] To explore this issue, we incorporate heterogenous households into the Schumpeterian model with endogenous takeoff in Peretto (2015). Then, we derive the evolution of income inequality when the economy experiences an endogenous transition from stagnation to growth. Our findings can be summarized as follows.

The model initially features a pre-industrial era, in which the economy is in stagnation with very slow economic growth. In this pre-industrial era, income inequality is determined by an unequal distribution of land ownership and remains stationary. When the market size becomes sufficiently large due to population growth, the economy begins to experience innovation, and the output growth rate becomes gradually rising over time until it converges to the steady state. In this industrial era, income inequality also gradually rises until the economy reaches the balanced growth path.

The intuition of the above results can be explained as follows. Our model features heterogeneous households in the form of an unequal distribution of assets. This wealth inequality gives rise to income inequality.[2] In the pre-industrial era, income inequality is caused by an unequal distribution of land ownership, which is stationary over time and leads to a constant degree of income inequality. In the industrial era, the economy experiences innovation and a gradually rising growth rate, which in turn leads to a gradually rising interest rate through the households' Euler equation. As a result, the importance of asset income, which is unequally distributed, increases over time and gives rise to increasing inequality until the economy reaches the balanced growth path.

We also calibrate the model to current data in the UK to perform a quantitative analysis. Simulating the transitional paths of the output growth rate and the real interest rate, we find that the increase in the simulated growth rate and the simulated interest rate is consistent with historical data in the UK. Finally, we simulate the transitional path of income inequality and find that income inequality increases sharply when the takeoff occurs. When the economy reaches the balanced growth path, income inequality is almost twice as high as the level prior to the takeoff, and the steady-state level of income inequality is in line with the Gini coefficient of income in the UK in recent time.

This study relates to the literature on innovation and economic growth. The seminal R&D-based growth model in Romer (1990) features the invention of new products (i.e., horizontal innovation) as the engine of growth. Aghion and Howitt (1992) develop the Schumpeterian model, in which economic growth is driven by the development of higher-quality products (i.e., vertical innovation).[3] Subsequent studies, such as Smulders and van de Klundert (1995), Peretto (1998, 1999) and Howitt (1999), combine vertical and horizontal innovation that gives rise to the second-generation Schumpeterian model.[4] This study con-

[1] Lindert (2000a, b) also finds a rise in income inequality in Britain in as early as the late 18th century.

[2] Piketty (2014) provides evidence for the importance of wealth inequality on income inequality.

[3] See also Grossman and Helpman (1991) and Segerstrom *et al.* (1990).

[4] Laincz and Peretto (2006), Ha and Howitt (2007), Madsen (2008, 2010) and Ang and Madsen (2011) provide supportive empirical evidence for the second-generation Schumpeterian model.

tributes to the literature by introducing heterogeneous households into a second-generation Schumpeterian model with endogenous takeoff to explore the effects of innovation on income inequality.[5]

This study also relates to the literature on inequality and economic growth. Early studies in this literature explore how inequality affects economic growth via capital accumulation; see for example Galor and Zeira (1993) and Aghion and Bolton (1997). An interesting study by Galor and Moav (2004) shows that in the early (later) stage of development in which the accumulation of physical (human) capital is the main engine of growth, inequality stimulates (stifles) economic growth. Subsequent studies consider how inequality affects the demand and supply of R&D in the innovation-driven growth model. On the demand side, Zweimuller (2000) and Foellmi and Zweimuller (2006) find that inequality has both a positive price effect and a negative market-size effect on the demand for innovation due to households' non-homothetic preferences. On the supply side, Garcia-Penalosa and Wen (2008) explore the insurance effect of redistribution, which increases innovation by providing more incentives for risk-averse agents to become R&D entrepreneurs.[6] Recent studies by Jones and Kim (2018) and Aghion *et al.* (2019) focus on the relationship between innovation and top-income inequality. Other studies, such as Chu (2010), Chu and Cozzi (2018) and Chu *et al.* (2019), analyze the effects of patent policy and monetary policy on innovation and income inequality. This study differs from previous studies by considering a Schumpeterian model with endogenous takeoff and analyzing the historical evolution of income inequality from stagnation to growth.

Finally, this study relates to the literature on endogenous takeoff and economic growth. Galor and Weil (2000) and Galor and Moav (2002) provide the seminal studies and develop unified growth theory.[7] Unified growth theory explores how the quality-quantity tradeoff in childrearing and human capital accumulation allow an economy to escape from the Malthusian trap and experience economic growth.[8] Although the Schumpeterian model in Peretto (2015) features exogenous population growth and does not capture the Malthusian trap, the innovation-driven takeoff in the model relates to the Industrial Revolution, which is arguably the most important economic takeoff in human history.[9] Furthermore, this growth-theoretic framework allows us to explore how innovation affects the rate of return on assets and the evolution of income inequality upon incorporating heterogeneous households into the model.

The rest of this study is organized as follows. Section 2 presents the model. Section 3 analyzes the dynamics and derives the evolution of income inequality. Section 4 performs a quantitative analysis. Section 5 concludes.

---

[5] Aghion *et al.* (2019) and Madsen *et al.* (2018) provide empirical evidence that innovation and inequality have a positive relationship.

[6] Chou and Talmain (1996) also explore how redistribution affects the supply of labor for R&D.

[7] See also Jones (2001) and Hansen and Prescott (2002) for other early studies on endogenous takeoff.

[8] See Galor and Mountford (2008) and Ashraf and Galor (2011) for recent studies and empirical evidence that supports unified growth theory. Galor (2011) provides an excellent review of unified growth theory.

[9] Mokry (2016) writes that "innovations in Europe triggered the Industrial Revolution and the sustained economic progress that spread across the globe."

2 A Schumpeterian growth model with heterogeneous households and endogenous takeoff

The Schumpeterian model with endogenous takeoff is based on Peretto (2015). We introduce heterogeneous households into the Peretto model.[10] Our analysis provides a complete closed-form solution for economic growth and income inequality from stagnation to takeoff and eventually to the balanced growth path.

2.1 Heterogeneous households

There is a unit continuum of households, which are indexed by $h \in [0, 1]$. They have identical homothetic preferences over consumption. However, households are heterogeneous in their levels of wealth. Household h's utility function is given by

$$U(h) = \int_0^\infty e^{-\rho t} \ln c_t(h) dt, \tag{1}$$

where the parameter $\rho > 0$ is the rate of subjective discounting and $c_t(h)$ is household h's consumption of final good (numeraire). Household h maximizes (1) subject to

$$\dot{a}_t(h) = r_t a_t(h) + w_t L_t - c_t(h). \tag{2}$$

$a_t(h)$ is the real value of assets owned by household h, and r_t is the real interest rate. These assets include the ownership of land $R_t(h)$ and monopolistic firms. Household h supplies L_t units of labor to earn a real wage rate w_t, where L_t increases at the rate $\lambda > 0$. From standard dynamic optimization, the familiar Euler equation is

$$\frac{\dot{c}_t(h)}{c_t(h)} = r_t - \rho, \tag{3}$$

which shows that the growth rate of consumption is the same across households such that $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t = r_t - \rho$, where $c_t \equiv \int_0^1 c_t(h) dh$ is aggregate consumption.

2.2 Final good

Competitive firms produce final good Y_t using the following production function:

$$Y_t = \int_0^{N_t} X_t^\theta(i)[Z_t^\alpha(i) Z_t^{1-\alpha} L_t^\gamma(i) R_t^{1-\gamma}]^{1-\theta} di,$$

where $\{\theta, \gamma, \alpha\} \in (0, 1)$. $X_t(i)$ denotes the quantity of non-durable intermediate good $i \in [0, N_t]$, and N_t is the mass of available intermediate goods at time t. The productivity of

[10] We use the approach in Chu (2010), Chu and Cozzi (2018) and Chu *et al.* (2019) to model heterogeneous households in the Schumpeterian model.

intermediate good $X_t(i)$ depends on its own quality $Z_t(i)$ and also on the average quality $Z_t \equiv \frac{1}{N_t}\int_0^{N_t} Z_t(i)di$ of all intermediate goods capturing technology spillovers. The private return to quality is determined by α, and the degree of technology spillovers is determined by $1-\alpha$. Final-good firms also rent land R_t and recruit labor $L_t(i)$ for $i \in [0, N_t]$.

Profit maximization yields the following conditional demand functions $\{R_t^d, L_t^d(i), X_t^d(i)\}$:

$$R_t^d = (1-\gamma)(1-\theta)Y_t/\omega_t, \tag{4}$$

$$L_t^d(i) = \left\{\frac{\gamma(1-\theta)}{w_t} X_t^\theta(i)[Z_t^\alpha(i)Z_t^{1-\alpha}R_t^{1-\gamma}]^{1-\theta}\right\}^{1/[1-\gamma(1-\theta)]}, \tag{5}$$

$$X_t^d(i) = \left(\frac{\theta}{p_t(i)}\right)^{1/(1-\theta)} Z_t^\alpha(i)Z_t^{1-\alpha}L_t^\gamma(i)R_t^{1-\gamma}, \tag{6}$$

where $p_t(i)$ is the price of $X_t(i)$ and ω_t is the rental price of $R_t = R$, which is in fixed supply. $L_t(i)$ in (6) is the equilibrium level of labor in industry $i \in [0, N_t]$. Competitive producers of final good pay $\theta Y_t = \int_0^{N_t} p_t(i)X_t(i)di$ for intermediate goods and $\gamma(1-\theta)Y_t = w_t\int_0^{N_t} L_t(i)di$ for labor.

2.3 Intermediate goods and in-house R&D

Monopolistic firms produce differentiated intermediate goods with a linear technology that requires $X_t(i)$ units of final good to produce $X_t(i)$ units of intermediate good $i \in [0, N_t]$; therefore, the marginal cost of production for the monopolistic firm is one. The firm in industry i also incurs $\phi Z_t^\alpha(i)Z_t^{1-\alpha}$ units of final good as a fixed operating cost. To improve the quality of its product, the firm devotes $I_t(i)$ units of final good to R&D. The innovation specification is given by

$$\dot{Z}_t(i) = I_t(i). \tag{7}$$

In industry i, the monopolistic firm's (before-R&D) profit flow at time t is

$$\Pi_t(i) = [p_t(i) - 1]X_t(i) - \phi Z_t^\alpha(i)Z_t^{1-\alpha}. \tag{8}$$

The value of the monopolistic firm in industry i is

$$V_t(i) = \int_t^\infty \exp\left(-\int_t^s r_u du\right)[\Pi_s(i) - I_s(i)]\, ds. \tag{9}$$

The monopolistic firm in industry i maximizes (9) subject to (6)-(8) taking the equilibrium level of labor in the industry as given. The current-value Hamiltonian for this optimization problem is

$$H_t(i) = \Pi_t(i) - I_t(i) + \eta_t(i)\dot{Z}_t(i), \tag{10}$$

where $\eta_t(i)$ is the co-state variable on (7). We solve this optimization problem in the Appendix and derive the unconstrained profit-maximizing markup ratio given by $1/\theta$.

Competitive firms can also produce $X_t(i)$ with the same quality $Z_t(i)$ as the monopolistic firm.[11] However, they have a higher unit cost of production given by $\mu > 1$. Therefore, the equilibrium price chosen by the monopolistic firm becomes

$$p_t(i) = \min\{\mu, 1/\theta\}. \tag{11}$$

We assume $\mu < 1/\theta$ implying that $p_t(i) = \mu$.

We follow previous studies to consider a symmetric equilibrium in which $Z_t(i) = Z_t$ for $i \in [0, N_t]$. In this case, the market-clearing condition for labor implies $L_t(i) = L_t/N_t$, and the size of intermediate-good firms is also identical across all industries, such that $X_t(i) = X_t$.[12] From (6) and $p_t(i) = \mu$, the quality-adjusted firm size is

$$\frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \left(\frac{L_t}{N_t}\right)^{\gamma} R^{1-\gamma}. \tag{12}$$

We define the following transformed variable:

$$x_t \equiv \mu^{1/(1-\theta)} \frac{X_t}{Z_t} = \theta^{1/(1-\theta)} \left(\frac{L_t}{N_t}\right)^{\gamma} R^{1-\gamma}. \tag{13}$$

x_t is a state variable that is determined by the quality-adjusted firm size, which in turn depends on L_t/N_t.[13] Lemma 1 derives the rate of return on quality-improving R&D, which is increasing in the firm size x_t.

Lemma 1 *The rate of return to in-house R&D is given by*

$$r_t^q = \alpha \frac{\Pi_t}{Z_t} = \alpha \left[\frac{\mu - 1}{\mu^{1/(1-\theta)}} x_t - \phi\right]. \tag{14}$$

Proof. See the Appendix. ■

2.4 Entrants

Following previous studies, we assume that entrants have access to aggregate technology Z_t to ensure symmetric equilibrium at any time t. A new firm pays βX_t units of final good to set up its operation and enter the market with a new variety of differentiated products. $\beta > 0$ is a cost parameter, and the cost function βX_t captures the case in which the setup cost is increasing in the initial output volume of the firm. The asset-pricing equation determines the rate of return on assets as

$$r_t = \frac{\Pi_t - I_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{15}$$

[11] Here we assume diffusion of technologies from the monopolistic firm to competitive firms in each industry.
[12] Symmetry also implies $\Pi_t(i) = \Pi_t$, $I_t(i) = I_t$ and $V_t(i) = V_t$.
[13] See Laincz and Peretto (2006) for empirical evidence on N_t being proportional to L_t.

When entry is positive, the entry condition is given by

$$V_t = \beta X_t. \tag{16}$$

Substituting (7), (8), (13), (16) and $p_t(i) = \mu$ into (15) yields the return on entry as

$$r_t^e = \frac{\mu^{1/(1-\theta)}}{\beta}\left[\frac{\mu - 1}{\mu^{1/(1-\theta)}} - \frac{\phi + z_t}{x_t}\right] + \frac{\dot{x}_t}{x_t} + z_t, \tag{17}$$

where $z_t \equiv \dot{Z}_t/Z_t$ is the growth rate of aggregate quality.

2.5 Value of land

Let v_t denote the unit value of land. Then, the asset-pricing equation for v_t is $r_t v_t = \omega_t + \dot{v}_t$. This asset-pricing equation states that the return on land is determined by the rental price of land and the capital gain in land value.

2.6 General equilibrium

The equilibrium is a time path of allocations $\{a_t, c_t, Y_t, L_t, R_t, X_t(i), I_t(i)\}$ and a time path of prices $\{r_t, w_t, \omega_t, v_t, p_t(i), V_t(i)\}$ such that the following conditions are satisfied:

- households maximize utility taking $\{r_t, w_t, \omega_t\}$ as given;
- final-good firms produce Y_t and maximize profit taking $\{p_t(i), w_t, \omega_t\}$ as given;
- intermediate-good firms produce $X_t(i)$ and choose $\{p_t(i), I_t(i)\}$ to maximize $V_t(i)$ taking r_t as given;
- entrants make entry decisions taking V_t as given;
- the value of land and monopolistic firms adds up to the value of the households' assets such that $v_t R + V_t N_t = \int_0^1 a_t(h)dh \equiv a_t$;
- the market-clearing condition of land holds such that $\int_0^1 R_t(h)dh = R$;
- the market-clearing condition of labor holds such that $\int_0^{N_t} L_t(i)di = N_t L_t(i) = L_t$;
- the following market-clearing condition of final good also holds:

$$Y_t = c_t + N_t(X_t + \phi Z_t + I_t) + \dot{N}_t \beta X_t. \tag{18}$$

2.7 Aggregation

Substituting (6) into the production function and imposing symmetry yield

$$Y_t = (\theta/\mu)^{\theta/(1-\theta)} N_t^{1-\gamma} Z_t L_t^{\gamma} R^{1-\gamma}, \tag{19}$$

which also uses markup pricing $p_t(i) = \mu$. Therefore, the growth rate of output is

$$\frac{\dot{Y}_t}{Y_t} = (1-\gamma)n_t + z_t + \gamma\lambda, \tag{20}$$

which is determined by the variety growth rate $n_t \equiv \dot{N}_t/N_t$ and the quality growth rate z_t.

3 Dynamics

This section analyzes the dynamics of the model. See Section 3.1 for the dynamics of the aggregate economy. See Section 3.2 for the dynamics of the wealth distribution. See Section 3.3 for the dynamics of the income distribution.

3.1 Dynamics of the aggregate economy

We analyze the dynamics of the economy across three eras. In the pre-industrial era, neither variety innovation nor quality innovation is activated. In the first industrial era, variety innovation is activated. In the second industrial era, quality innovation is also activated.

In the pre-industrial era, the firm size x_t is so small (i.e., $x_t < \phi\mu^{1/(1-\theta)}/(\mu-1)$) that monopolistic firms cannot earn a positive profit. Therefore, all intermediate goods N_0 are produced by competitive firms at $p_t(i) = \mu$ given that they do not incur the operating cost. In this case, the intermediate-good sector generates zero profit, and the value of monopolistic firms is zero. Therefore, consumption is given by the rental income from land and the wage income from labor such that $c_t = \omega_t R + w_t L_t$, so the consumption-output ratio is simply $c_t/Y_t = 1-\theta$ in the pre-industrial era.

In the first industrial era, variety innovation is activated. In this case, the consumption-output ratio c_t/Y_t jumps to the steady-state value in Lemma 2 and remains at this value also in the second industrial era.

Lemma 2 *Once variety innovation is activated, the consumption-output ratio jumps to*

$$\frac{c_t}{Y_t} = \frac{\rho\beta\theta}{\mu} + 1 - \theta. \tag{21}$$

Proof. See the Appendix. ■

The above analysis implies that consumption and output grow at the same rate given by

$$g_t \equiv \frac{\dot{Y}_t}{Y_t} = \frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{22}$$

where the last equality uses (3). In the pre-industrial era, the growth rate of output is simply $g_t = \gamma\lambda$; therefore, the real interest rate is given by $r_t = \rho + \gamma\lambda$. In the first industrial era, the growth rate of output becomes $g_t = \gamma\lambda + (1-\gamma)n_t$ due to variety innovation; in this case, the real interest rate is given by $r_t = \rho + \gamma\lambda + (1-\gamma)n_t$. In the second industrial era, we can substitute (14) into (22) to derive the growth rate of output as

$$g_t = \gamma\lambda + (1-\gamma)n_t + z_t = \alpha\left[\frac{\mu - 1}{\mu^{1/(1-\theta)}}x_t - \phi\right] - \rho, \tag{23}$$

and the real interest rate is $r_t = \rho + g_t$. Then, (23) implies that the quality growth rate is

$$z_t = \alpha\left[\frac{\mu - 1}{\mu^{1/(1-\theta)}}x_t - \phi\right] - \rho - \gamma\lambda - (1-\gamma)n_t. \tag{24}$$

The quality growth rate z_t is positive if and only if $x_t > x_Z$, where x_Z is a threshold for the firm size x_t above which quality innovation starts to occur.[14] Intuitively, innovation requires the firm size x_t to be large enough so that it is profitable for firms to do in-house R&D.

From (13), the growth rate of the firm size x_t is given by

$$\frac{\dot{x}_t}{x_t} = \gamma(\lambda - n_t). \tag{25}$$

In the pre-industrial era, the growth rate of x_t is $\dot{x}_t/x_t = \gamma\lambda$ because $n_t = 0$. In the first industrial era, the growth rate of variety is given by

$$n_t = \frac{\mu^{1/(1-\theta)}}{\beta}\left[\frac{\mu - 1}{\mu^{1/(1-\theta)}} - \frac{\phi}{x_t}\right] - \rho, \tag{26a}$$

which is obtained from substituting (20), (22) and (25) into (17). The variety growth rate n_t is positive if and only if $x_t > \phi\mu^{1/(1-\theta)}/(\mu - 1 - \beta\rho) \equiv x_N$, where x_N is a threshold for the firm size x_t above which variety innovation starts to occur.[15] Substituting (26a) into (25) yields an one-dimensional differential equation in x_t in the first industrial era. In the second industrial era, the growth rate of variety becomes

$$n_t = \frac{\mu^{1/(1-\theta)}}{\beta}\left[\frac{\mu - 1}{\mu^{1/(1-\theta)}} - \frac{\phi + z_t}{x_t}\right] - \rho, \tag{26b}$$

where z_t is given in (24). Substituting (24) and (26b) into (25) also yields an one-dimensional differential equation in x_t in the second industrial era. Therefore, the dynamics of x_t in (25) is autonomous in all three eras. Proposition 1 summarizes the dynamics of x_t.

[14] We provide the definition of x_Z in the proof of Proposition 1.

[15] We assume $x_N < x_Z$ (i.e., variety innovation occurs before quality innovation). When $\phi\mu^{1/(1-\theta)}/(\mu-1) < x_t < x_N$, monopolistic firms in N_0 are able to earn positive profits, but we assume that intermediate goods are produced by competitive firms until x_t reaches x_N because it takes time for the sector to be monopolized.

Proposition 1 *In the pre-industrial era (i.e., $x_t < x_N$), the dynamics of x_t is given by $\dot{x}_t = \gamma\lambda x_t$. In the first industrial era (i.e., $x_t \in [x_N, x_Z]$), the dynamics of x_t is given by*

$$\dot{x}_t = \gamma\left[\frac{\mu^{1/(1-\theta)}}{\beta}\phi - \left(\frac{\mu-1}{\beta} - \lambda - \rho\right)x_t\right].$$

In the second industrial era (i.e., $x_t > x_Z$), the linearized dynamics of x_t is given by

$$\dot{x}_t = \gamma\left\{\frac{\mu^{1/(1-\theta)}}{\beta}\left[(1-\alpha)\phi - \lambda - \rho\right] - \left[(1-\alpha)\frac{\mu-1}{\beta} - \lambda - \rho\right]x_t\right\}.$$

Therefore, given $\rho + \lambda < \min\{(1-\alpha)\phi, (1-\alpha)(\mu-1)/\beta\}$, the dynamics of x_t is stable, and x_t converges to a unique steady state. The steady-state values $\{x^*, g^*\}$ are

$$x^* = \mu^{1/(1-\theta)}\frac{(1-\alpha)\phi - (\rho+\lambda)}{(1-\alpha)(\mu-1) - \beta(\rho+\lambda)} > x_Z, \tag{27}$$

$$g^* = \alpha\left[\frac{\mu-1}{\mu^{1/(1-\theta)}}x^* - \phi\right] - \rho > 0. \tag{28}$$

Proof. See the Appendix. ■

The dynamics of x_t in (25) and (26) shows that given an initial value x_0, the state variable x_t grows over time. When the firm size x_t becomes sufficiently large, variety innovation occurs, and then quality innovation also occurs. Eventually, x_t converges to its steady-state value x^* in (27), which also determines $N_t^* = [\theta^{1/(1-\theta)}R^{1-\gamma}/x^*]^{1/\gamma}L_t$ that grows at the rate λ on the balanced growth path. Figure 1 illustrates the dynamics of x_t.[16]

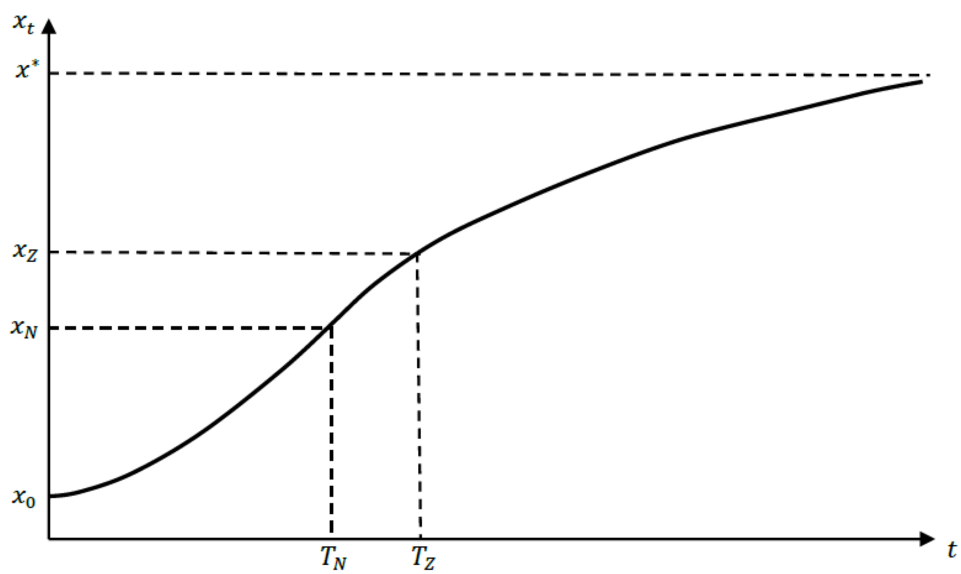

Figure 1: Transition path of the firm size

[16] T_N (T_Z) denotes the time when variety (quality) innovation is activated.

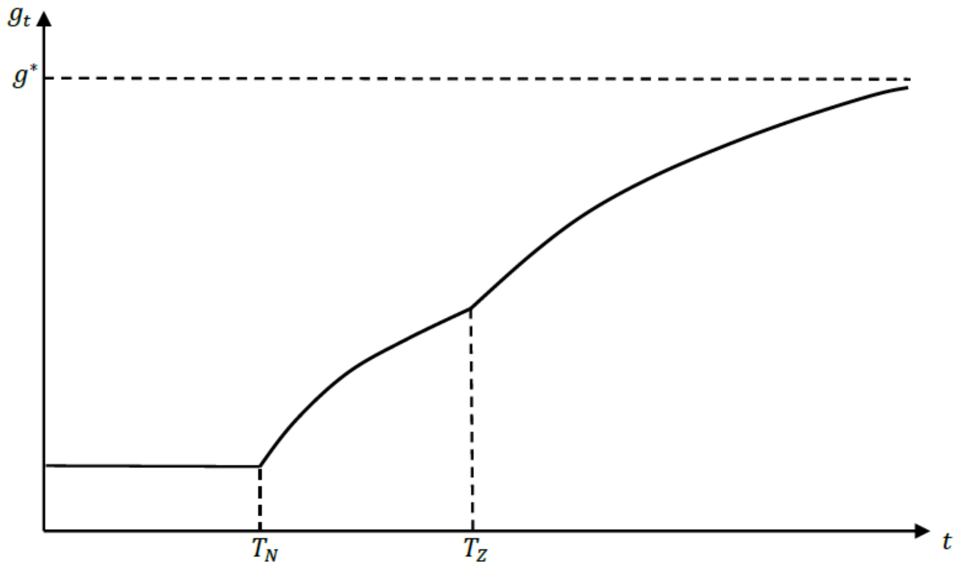

Figure 2: Transition path of the growth rate

Figure 2 presents the dynamics of economic growth. In the pre-industrial era, the growth rate of output is simply $g_t = \gamma\lambda$ due to the absence of innovation. In the first industrial era, the firm size x_t determines the variety growth rate according to (26a) and the output growth rate according to $g_t = \gamma\lambda + (1-\gamma)n_t$. In the second industrial era, the firm size x_t determines the quality growth rate and the output growth rate g_t according to (23). When x_t converges to x^* in (27), g_t also converges to its steady-state value g^* in (28). This gradual acceleration in economic growth in Figure 2 is consistent with historical data in the UK. Figure 3 plots the log level of real GDP,[17] in which the slope shows the growth rate. The average growth rates in the UK were 0.71% in the first half of the 18th century, 1.24% in the second half of the 18th century, 1.86% in the first half of the 19th century, 2.23% in the second half of the 19th century, 1.50% in the first half of the 20th century and 2.55% from the second half of the 20th century onwards. Except for the wartime periods in the first half of the 20th century, the UK economy experiences a gradually rising growth rate as in our Schumpeterian model of endogenous takeoff.

[17] Data source: Federal Reserve Bank of St. Louis.

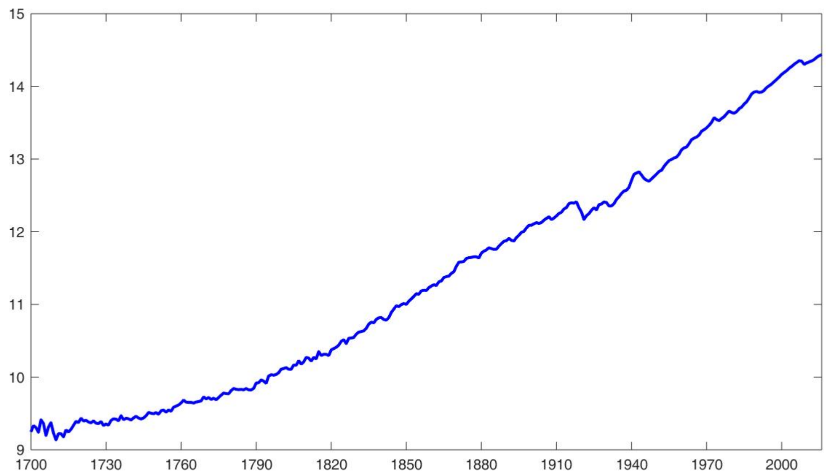

Figure 3: Log of real GDP in the UK from 1700 to 2016

3.2 Dynamics of the wealth distribution

In the pre-industrial era, the value of monopolistic firms is zero; therefore, the wealth distribution is determined solely by the distribution of land. The initial share of land owned by household h is $s_R(h) \equiv R_0(h)/R$, which is exogenously given at time 0. We find that during the pre-industrial era, the distribution of land is stationary. When the economy reaches the first industrial era at time $\tau > 0$, the value of monopolistic firms becomes positive. Then, for any given x_t at any time $t \geq \tau$, we find that the distribution of assets, which consist of land and intangible capital, continues to be stationary and determined by the initial distribution of land at time 0. It is useful to recall that the aggregate economy features transition dynamics determined by the evolution of x_t. However, the wealth distribution is stationary despite the transition dynamics in the aggregate economy because the consumption-output ratio is stationary as shown in Lemma 2, which in turn implies that the consumption-wealth ratio is also stationary.

Aggregating (2) across all households yields the following asset-accumulation equation:

$$\dot{a}_t = r_t a_t + w_t L_t - c_t. \tag{29}$$

We substitute (3) and $\gamma(1-\theta)Y_t = w_t L_t$ into (29) to derive the following differential equation for c_t/a_t:

$$\frac{\dot{c}_t}{c_t} - \frac{\dot{a}_t}{a_t} = \left[1 - \gamma(1-\theta)\frac{Y_t}{c_t}\right]\frac{c_t}{a_t} - \rho, \tag{30}$$

where c_t/Y_t is simply $1-\theta$ in the pre-industrial era and given by (21) in the industrial era. Therefore, the coefficient on c_t/a_t is always positive implying that the consumption-wealth ratio must jump to

$$\frac{c_t}{a_t} = \frac{\rho}{1 - \gamma(1-\theta)Y_t/c_t}, \tag{31}$$

whenever the consumption-output ratio c_t/Y_t changes.

Let $s_{a,t}(h) \equiv a_t(h)/a_t$ denote the share of assets owned by household h. Then, the growth rate of $s_{a,t}(h)$ is given by

$$\frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} = \frac{\dot{a}_t(h)}{a_t(h)} - \frac{\dot{a}_t}{a_t} = \frac{c_t - w_t L_t}{a_t} - \frac{s_{c,t}(h)c_t - w_t L_t}{a_t(h)}, \tag{32}$$

where $s_{c,t}(h) \equiv c_t(h)/c_t$ is the share of consumption by household h at time t. Substituting $\gamma(1-\theta)Y_t = w_t L_t$ and (31) into (32) yields

$$\dot{s}_{a,t}(h) = \rho s_{a,t}(h) - \frac{c_t}{a_t}\left[s_{c,t}(h) - 1\right] - \rho. \tag{33}$$

To achieve stability of $s_{a,t}(h)$, $\dot{s}_{a,t}(h) = 0$ must hold for any $t \geq 0$ because $s_{a,t}(h)$ is a pre-determined variable and its coefficient is positive. Therefore, we have $s_{a,t}(h) = s_{a,0}(h)$, which is achieved by $s_{c,t}(h)$ jumping to the steady-state values at $t = 0$ and $t = \tau$. Imposing $\dot{s}_{a,t}(h) = 0$ on (33) yields

$$s_{c,t}(h) = \frac{\rho}{c_t/a_t}[s_{a,0}(h) - 1] + 1, \tag{34}$$

where c_t/a_t is given in (31), in which c_t/Y_t is simply $1-\theta$ in the pre-industrial era (i.e., $t < \tau$) and given by (21) in the two industrial eras (i.e., $t \geq \tau$). Therefore, we have $s_{c,t}(h) = s_{c,0}(h)$ for $t < \tau$ and $s_{c,t}(h) = s_{c,\tau}(h)$ for $t \geq \tau$. At time 0, the share of assets owned by household h is determined by its share of land such that $s_{a,0}(h) = a_0(h)/a_0 = R_0(h)/R \equiv s_R(h)$. We summarize the dynamics of $s_{a,t}(h)$ in Proposition 2.

Proposition 2 *The dynamics of $s_{a,t}(h)$ is given by an one-dimensional differential equation:*

$$\dot{s}_{a,t}(h) = \rho[s_{a,t}(h) - s_R(h)]. \tag{35}$$

Also, the wealth distribution is stationary and determined by the initial distribution of land.

Proof. Proven in text. ■

Finally, we derive the Gini coefficient of wealth and show that it is also stationary. $a_t(h)$ is the share of wealth owned by household h, where the identity index $h \in [0,1]$ is ordered in an ascending order of wealth. Let $\sigma_{a,t}$ denote the Gini coefficient of wealth at time t, which is defined as

$$\sigma_{a,t} \equiv 1 - 2\int_0^1 \mathcal{L}_{a,t}(h)dh,$$

where the Lorenz curve of wealth is given by

$$\mathcal{L}_{a,t}(h) \equiv \frac{\int_0^h a_t(\chi)d\chi}{\int_0^1 a_t(\chi)d\chi} = \frac{\int_0^h a_t(\chi)d\chi}{a_t} = \int_0^h s_{a,t}(\chi)d\chi = \int_0^h s_{a,0}(\chi)d\chi = \int_0^h s_R(\chi)d\chi.$$

Therefore, the Gini coefficient of wealth is stationary, such that $\sigma_{a,t} = \sigma_{a,0} = \sigma_R$ for all t, where σ_R is the Gini coefficient of land ownership at time 0.

3.3 Dynamics of the income distribution

The income distribution is endogenous and nonstationary but analytically tractable. Although the wealth distribution is stationary, the transition dynamics in the aggregate economy gives rise to an endogenous evolution of the income distribution. However, once we derive the transitional path of the real interest rate, we can also derive the transitional path of income inequality.

Income received by household h is given by

$$y_t(h) = r_t a_t(h) + w_t L_t. \tag{36}$$

Aggregating (36) yields the aggregate level of income as

$$y_t = r_t a_t + w_t L_t. \tag{37}$$

Let $s_{y,t}(h) \equiv y_t(h)/y_t$ denote the share of income received by household h. Then, we have

$$s_{y,t}(h) = \frac{r_t a_t(h) + w_t L_t}{r_t a_t + w_t L_t} = \frac{r_t a_t}{r_t a_t + w_t L_t} s_R(h) + \frac{w_t L_t}{r_t a_t + w_t L_t}, \tag{38}$$

where we have used $s_{a,t}(h) = s_R(h)$. Therefore, the evolution of the share of income received by household h is determined by the evolution of the asset-wage income ratio $r_t a_t/(w_t L_t)$. An increase in the asset-wage income ratio $r_t a_t/(w_t L_t)$ would increase (decrease) the income share of household h if its wealth share $s_R(h)$ is larger (smaller) than one, which is the average wealth share.[18] In other words, an increase in the asset-wage income ratio enlarges the dispersion of income because wealth inequality drives income inequality in our model. Lemma 3 derives the Gini coefficient of income as our measure of income inequality $\sigma_{y,t}$, which is increasing in the asset-wage income ratio $r_t a_t/(w_t L_t)$.[19]

Lemma 3 *The Gini coefficient of income is given by*[20]

$$\sigma_{y,t} = \frac{r_t a_t}{r_t a_t + w_t L_t} \sigma_R. \tag{39}$$

Proof. See the Appendix. ■

In the pre-industrial era, the value of intangible assets is zero, and hence we have

$$r_t a_t = (\rho + g_t) a_t = \left(1 + \frac{\gamma\lambda}{\rho}\right) \rho a_t = \left(1 + \frac{\gamma\lambda}{\rho}\right) \omega_t R. \tag{40}$$

[18] Recall that there is a unit continuum of households.

[19] Madsen (2017) provides evidence that asset returns are an important determinant of income inequality.

[20] The coefficient of variation of income is also given by $\sigma_{y,t} = \frac{r_t a_t}{r_t a_t + w_t L_t} \sigma_R$ if we instead define $\sigma_{y,t} \equiv \sqrt{\int_0^1 [s_{y,t}(h) - 1]^2 dh}$ and $\sigma_R \equiv \sqrt{\int_0^1 [s_R(h) - 1]^2 dh}$ as the coefficients of variation of income and wealth.

Therefore, income inequality in the pre-industrial era (i.e., $t < \tau$) is simply

$$\sigma_{y,t} = \left(1 + \frac{\rho}{\rho + \gamma\lambda} \frac{w_t L_t}{\omega_t R}\right)^{-1} \sigma_R = \left(1 + \frac{\rho}{\rho + \gamma\lambda} \frac{\gamma}{1-\gamma}\right)^{-1} \sigma_R. \tag{41}$$

Proposition 3 derives $\sigma_{y,t}$ in the industrial eras. We define a parameter $\Theta \equiv \beta\theta/(1-\theta)$.

Proposition 3 *After variety innovation occurs, the degree of income inequality for $t \geq \tau$ is*

$$\sigma_{y,t} = \left(1 + \frac{\rho}{\rho + g_t} \frac{\gamma}{1 - \gamma + \rho\Theta/\mu}\right)^{-1} \sigma_R. \tag{42}$$

Proof. See the Appendix. ■

We summarize the evolution of income inequality as follows. In the pre-industrial era, income inequality is given in (41). In the first industrial era, the value of intangible assets becomes positive, and income inequality jumps to

$$\sigma_{y,t} = \left(1 + \frac{\rho}{\rho + \gamma\lambda + (1-\gamma)n_t} \frac{\gamma}{1 - \gamma + \rho\Theta/\mu}\right)^{-1} \sigma_R, \tag{43}$$

where the term $\rho\Theta/\mu$ captures the effect of the emergence of intangible assets. Then, income inequality increases over time due to the rising variety growth rate n_t in (26a). In the second industrial era, income inequality further increases to

$$\sigma_{y,t} = \left(1 + \frac{\rho}{\rho + \gamma\lambda + (1-\gamma)n_t + z_t} \frac{\gamma}{1 - \gamma + \rho\Theta/\mu}\right)^{-1} \sigma_R, \tag{44}$$

where $g_t = \gamma\lambda + (1-\gamma)n_t + z_t$ in (23) increases over time until reaching the balanced growth path. On the balanced growth path, income inequality σ_y^* is determined by the steady-state growth rate g^* in (28). Figure 4 summarizes the dynamics of $\sigma_{y,t}$ from stagnation to takeoff and eventually to the steady state.

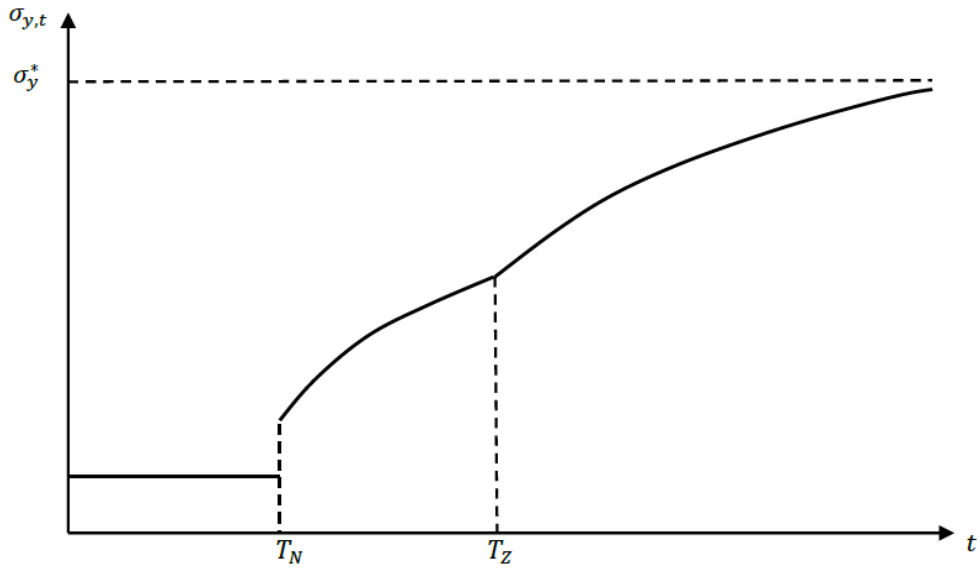

Figure 4: Transition path of income inequality

4 Quantitative analysis

In this section, we calibrate the model to UK data in order to perform a quantitative analysis. The model features the following parameters: $\{\rho, \alpha, \lambda, \theta, \beta, \gamma, \mu, \phi\}$. We set the discount rate ρ to 0.04. We follow Iacopetta *et al.* (2019) to set the degree of technology spillovers $1-\alpha$ to 0.833. In the UK, the long-run population growth rate λ is 0.6%.[21] Then, we calibrate the remaining parameters $\{\theta, \beta, \gamma, \mu, \phi\}$ by matching the following moments for the UK economy: 52.6% for labor income as a share of output,[22] 74.4% for consumption as a share of output,[23] 12.3% for housing rents as a share of output,[24] 2.5% for the growth rate of output,[25] and 18.4% for investment as a share of output.[26] Table 1 summarizes the calibrated parameter values.[27] These parameter values imply a rate of asset returns of 6.5% and R&D as a share of output of 2.0%, which are in line with UK data.

Table 1: Calibrated parameter values

ρ	α	λ	θ	β	γ	μ	ϕ
0.040	0.167	0.006	0.351	14.468	0.810	2.138	0.245

[21] Data source: Maddison Project Database.
[22] Data source: Office for National Statistics.
[23] Data source: Office for National Statistics.
[24] Data source: New Economics Foundation.
[25] Data source: Federal Reserve Bank of St. Louis.
[26] Data source: Office for National Statistics. To compute this moment from the model, we add up expenses on intermediate goods and horizontal/vertical R&D. One can think of the intermediate goods in our model as investment in capital that depreciates rapidly.
[27] The calibrated value of μ seems high but implies a reasonable profit share of output of 11.5%.

Figure 5 presents the simulated paths of the output growth rate and the real interest rate along with the HP-filter trends of the GDP growth rate and the rate of return on non-residential fixed capital in the UK.[28] We choose an initial value x_0 such that the takeoff occurs in the late 18th century.[29] This figure shows that the output growth rate increases from about 0.5% in the late 18th century to 2.5% in recent time. This gradual increase in the growth rate and the magnitude of the increase are in line with historical data in the UK. Figure 5 also shows that the real interest rate increases from 4.5% in the late 18th century to an average of 5.9% in the 19th century and reaches an average of 6.4% in the 20th century. The average rates of return on non-residential fixed capital in the UK were 5.1% in the 18th century, 6.0% in the 19th century, and 7.0% from the 20th century onwards.[30] Therefore, the increase in the rate of return on assets and the magnitude of the increase in asset returns predicted by our model are also in line with historical data.

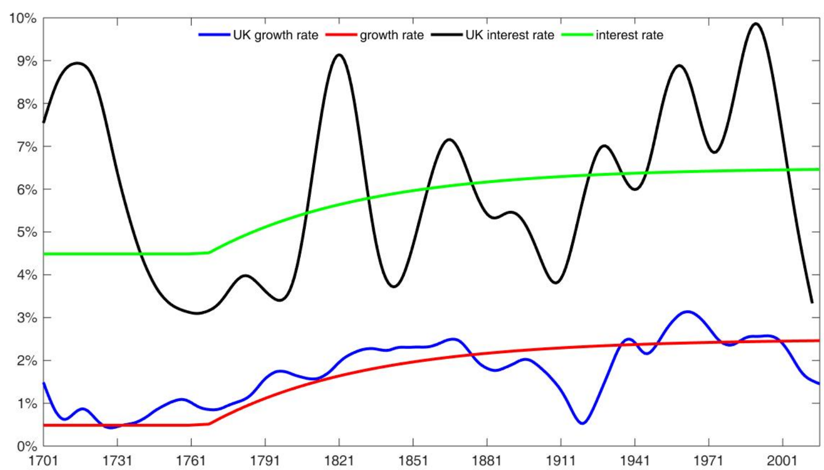

Figure 5: Simulated paths of the growth rate and the interest rate

The increase in the real interest rate in Figure 5 implies an increase in income inequality in our model. Figure 6 presents the simulated path of income inequality in terms of percent changes from its initial value prior to the takeoff. This figure shows that income inequality increases sharply by about 50% when the takeoff occurs. When the economy reaches the balanced growth path, income inequality would have almost doubled. Our model takes the degree of wealth inequality as given. If we consider a Gini coefficient of wealth of 0.732 in recent time,[31] then we can also simulate the Gini coefficient of income. Figure 7 reports the simulated path of income inequality along with the Gini coefficient of income in the UK from 1961 to 2017.[32] It shows that the simulated Gini coefficient of income increases from 0.15 before the takeoff to 0.29 in the steady state.

[28] Here we use a smoothing parameter of 1000 on the annual data in order to extract a smoother trend.

[29] According to Ashton (1998), the Industrial Revolution started in as early as 1760.

[30] See Madsen (2017). The authors are grateful to Jakob Madsen for sharing this data series.

[31] Data source: Credit Suisse Global Wealth Databook.

[32] Data source: Institute for Fiscal Studies. Data available from 1961.

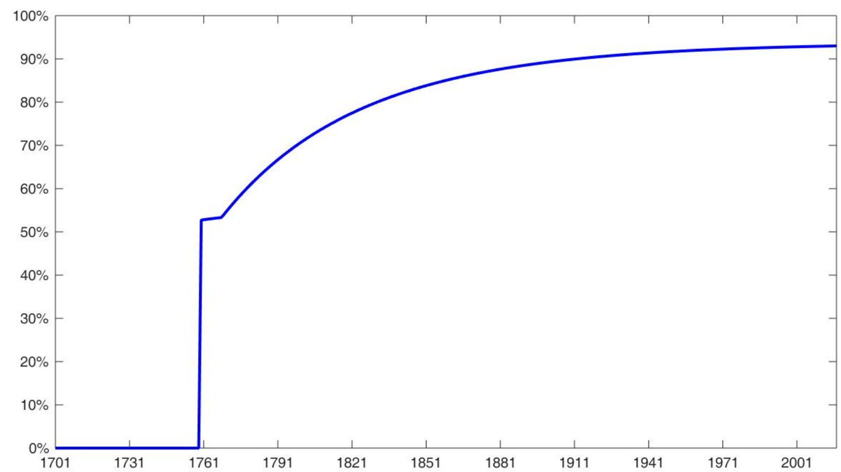

Figure 6: Simulated path of income inequality (percent change)

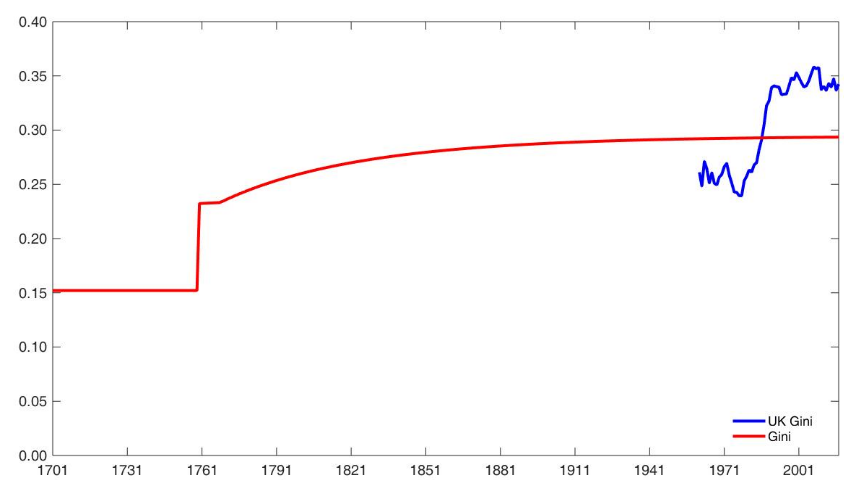

Figure 7: Simulated path of income inequality (Gini coefficient)

Williamson (1980, 1985) and Lindert (2000a, 2000b) examine historical data in Britain and document that income inequality, based on different measures, increases in the late 18th century/early 19th century and levels off after the mid-19th century. Then, income inequality, measured by the top 1% income share, decreases from the early 20th century to the late 1970's.[33] As for the Gini coefficient of income, it decreases from 0.27 in the early

[33] World Inequality Database documents a decrease in the top 1% income share from 20% in the early 20th century to 5% in the late 1970's.

1960's to 0.24 in the late 1970's before rising again to as high as 0.36 in recent time with an average value of 0.30 from 1961 to 2017 in the UK. Therefore, the long-run level of income inequality predicted by our model is in line with recent data in the UK. Furthermore, our model is able to deliver the pattern of rising income inequality in the late 18th century/early 19th century and its leveling off in the late 19th century. However, our model is unable to explain the decrease in income inequality from the early 20th century to the late 1970's. The reason is that this decrease in income equality is driven by a decrease in wealth inequality,[34] whereas our model takes wealth inequality as given.

To address this issue, we consider historical data on the income and wealth shares owned by the top households, which have longer time series than the Gini coefficient. In our model, the share of income owned by the top ε households is given by

$$\int_{1-\varepsilon}^{1} s_{y,t}(h)dh = \frac{r_t a_t}{r_t a_t + w_t L_t} \int_{1-\varepsilon}^{1} s_R(h)dh + \frac{w_t L_t}{r_t a_t + w_t L_t}\varepsilon, \tag{45}$$

where $\int_{1-\varepsilon}^{1} s_R(h)dh$ is the share of wealth owned by the top ε households. We use historical data on the top 10% wealth share in the UK along with the asset-wage income ratio $r_t a_t/(w_t L_t)$ computed from our model to simulate the top 10% income share. Figure 8 presents the simulated path of the top 10% income share along with data in the UK from 1900 to 2010.[35] Given the data on wealth inequality, our model now predicts that income inequality rises in the 19th century and falls from the early 20th century to the 1970's. After that, income inequality becomes rising again. This pattern matches the data. Furthermore, the average value of the top 10% income share in the UK from 1900 to 2010 is 0.37, whereas our model predicts an average value of 0.36 in this period.

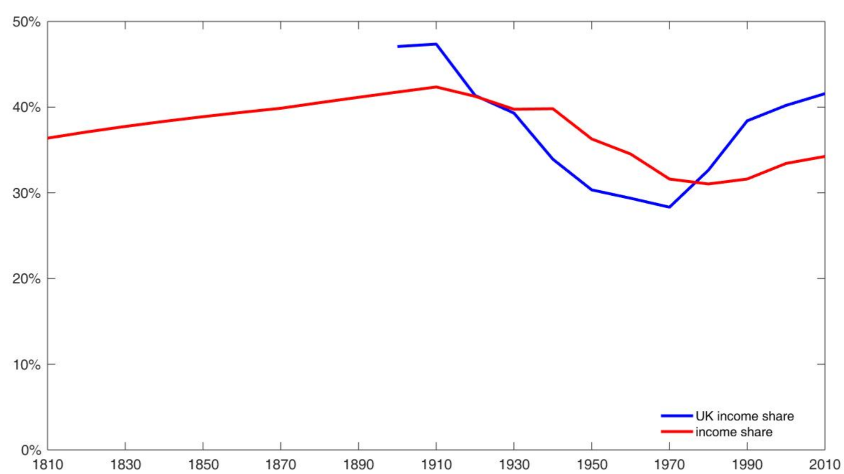

Figure 8: Simulated path of the top 10% income share

[34] World Inequality Database documents a decrease in the top 1% wealth share from 70% in the early 20th century to less than 20% in the early 1980's.

[35] Data source: Piketty (2014). Data on the top 10% wealth (income) share is available from 1810 (1900).

5 Conclusion

This study explores the historical origins of income inequality from stagnation to growth in a Schumpeterian model with endogenous takeoff and heterogeneous households. Our results can be summarized as follows. In the pre-industrial era, the economy is in stagnation, and income inequality is determined by an unequal distribution of land ownership and remains stationary. In the industrial era, income inequality rises sharply, and the gradually rising growth rate in the economy causes income inequality to increase further over time until the economy reaches the balanced growth path. We also calibrate the model to perform a quantitative analysis and find that the simulation results are roughly in line with historical data in the UK.

To keep the dynamics analytically tractable, we do not consider the accumulation of physical capital in our model. In the presence of physical capital, a no-arbitrage condition would imply that the rental price of capital is determined by the real interest rate, which in turn is the driving force for the evolution of income inequality in our model. Furthermore, we assume that households have homothetic preferences under which the income distribution does not affect the aggregate economy; in other words, changes in inequality have no direct effect on economic growth in our model. Given that previous studies have already explored in details how inequality could affect economic growth (see the discussion in the introduction), this study focuses on how the innovation-driven takeoff during the Industrial Revolution influences the evolution of income inequality. We leave to future research the interesting question of how inequality affects the takeoff of an economy.[36]

[36] An interesting study by Voigtlander and Voth (2006) shows that "redistributive institutions [in Britain] were not decisive in fostering industrialization."

References

[1] Aghion, P., Akcigit, U., Bergeaud, A., Blundell, R., and Hemous, D., 2019. Innovation and top income inequality. *Review of Economic Studies*, 86, 1-45.

[2] Aghion, P., and Bolton, P., 1997. A theory of trickle-down growth and development. *Review of Economic Studies*, 64, 151-172.

[3] Aghion, P., and Howitt, P., 1992. A model of growth through creative destruction. *Econometrica*, 60, 323-351.

[4] Ang, J., and Madsen, J., 2011. Can second-generation endogenous growth models explain the productivity trends and knowledge production in the Asian miracle economies?. *Review of Economics and Statistics*, 93, 1360-1373.

[5] Ashraf, Q., and Galor, O., 2011. Dynamics and stagnation in the Malthusian epoch. *American Economic Review*, 101, 2003-2041.

[6] Ashton, T. S., 1998. *The Industrial Revolution, 1760-1830*. Oxford University Press.

[7] Chou, C.-F., and Talmain, G., 1996. Redistribution and growth: Pareto improvements. *Journal of Economic Growth*, 1, 505-523.

[8] Chu, A., 2010. Effects of patent policy on income and consumption inequality in an R&D-based growth model. *Southern Economic Journal*, 77, 336-350.

[9] Chu, A., and Cozzi, G., 2018. Effects of patents versus R&D subsidies on income inequality. *Review of Economic Dynamics*, 29, 68-84.

[10] Chu, A., Cozzi, G., Fan, H., Furukawa, Y., and Liao, C., 2019. Innovation and inequality in a monetary Schumpeterian model with heterogeneous households and firms. *Review of Economic Dynamics*, 34, 141-164.

[11] Foellmi, R., and Zweimuller, J., 2006. Income distribution and demand-induced innovations. *Review of Economic Studies*, 73, 941-960.

[12] Galor, O., 2011. *Unified Growth Theory*. Princeton University Press.

[13] Galor, O., and Moav, O., 2002. Natural selection and the origin of economic growth. *Quarterly Journal of Economics*, 117, 1133-1192.

[14] Galor, O., and Moav, O., 2004. From physical to human capital accumulation: inequality and the process of development. *Review of Economic Studies*, 71, 1001-1026.

[15] Galor, O., and Mountford, A., 2008. Trading population for productivity: Theory and evidence. *Review of Economic Studies*, 75, 1143-1179.

[16] Galor, O., and Weil, D., 2000. Population, technology and growth: From the Malthusian regime to the demographic transition. *American Economic Review*, 110, 806-828.

[17] Galor, O., and Zeira, J., 1993. Income distribution and macroeconomics. *Review of Economic Studies*, 60, 35-52.

[18] Garcia-Penalosa, C., and Wen, J.-F., 2008. Redistribution and entrepreneurship with Schumpeterian growth. *Journal of Economic Growth*, 13, 57-80.

[19] Grossman, G., and Helpman, E., 1991. Quality ladders in the theory of growth. *Review of Economic Studies*, 58, 43-61.

[20] Ha, J., and Howitt, P., 2007. Accounting for trends in productivity and R&D: A Schumpeterian critique of semi-endogenous growth theory. *Journal of Money, Credit, and Banking*, 33, 733-774.

[21] Hansen, G., and Prescott, E., 2002. Malthus to Solow. *American Economic Review*, 2002, 92, 1205-1217.

[22] Howitt, P., 1999. Steady endogenous growth with population and R&D inputs growing. *Journal of Political Economy*, 107, 715-730.

[23] Iacopetta, M., Minetti, R., and Peretto, P., 2019. Financial markets, industry dynamics and growth. *Economic Journal*, forthcoming.

[24] Jones, C., 2001. Was an industrial revolution inevitable? Economic growth over the very long run. *B.E. Journal of Macroeconomics (Advances)*, 1, 1-45.

[25] Jones, C., and Kim, J., 2018. A Schumpeterian model of top income inequality. *Journal of Political Economy*, 126, 1785-1826.

[26] Kuznets, S., 1955. Economic growth and income inequality. *American Economic Review*, 45, 1-28.

[27] Laincz, C., and Peretto, P., 2006. Scale effects in endogenous growth theory: An error of aggregation not specification. *Journal of Economic Growth*, 11, 263-288.

[28] Lindert, P., 2000a. Three centuries of inequality in Britain and America. *Handbook of Income Distribution*, 1, 167-216.

[29] Lindert, P., 2000b. When did inequality rise in Britain and America?. *Journal of Income Distribution*, 9, 11-25.

[30] Madsen, J., 2008. Semi-endogenous versus Schumpeterian growth models: Testing the knowledge production function using international data. *Journal of Economic Growth*, 13, 1-26.

[31] Madsen, J., 2010. The anatomy of growth in the OECD since 1870. *Journal of Monetary Economics*, 57, 753-767.

[32] Madsen, J., 2017. Is inequality increasing in $r - g$? Piketty's principle of capitalist economics and the dynamics of inequality in Britain, 1210-2013. CAMA Working Paper No. 63/2017.

[33] Madsen, J., Minniti, A., and Venturini, F., 2018. Wealth inequality in the long run: A Schumpeterian growth perspective. CAMA Working Paper No. 35/2018.

[34] Mokyr, J., 2016. *A Culture of Growth: The Origins of the Modern Economy*. Princeton University Press.

[35] Peretto, P., 1998. Technological change and population growth. *Journal of Economic Growth*, 3, 283-311.

[36] Peretto, P., 1999. Cost reduction, entry, and the interdependence of market structure and economic growth. *Journal of Monetary Economics*, 43, 173-195.

[37] Peretto, P., 2015. From Smith to Schumpeter: A theory of take-off and convergence to sustained growth. *European Economic Review*, 78, 1-26.

[38] Piketty, T., 2014. *Capital in the Twenty-First Century*. Harvard University Press.

[39] Romer, P., 1990. Endogenous technological change. *Journal of Political Economy*, 98, S71-S102.

[40] Segerstrom, P., Anant, T., and Dinopoulos, E., 1990. A Schumpeterian model of the product life cycle. *American Economic Review*, 80, 1077-91.

[41] Voigtlander, N., and Voth, H.-J., 2006. Why England? Demographic factors, structural change and physical capital accumulation during the Industrial Revolution. *Journal of Economic Growth*, 11, 319-361.

[42] Williamson, J., 1980. Earnings inequality in nineteenth-century Britain. *Journal of Economic History*, 40, 457-475.

[43] Williamson, J., 1985. *Did British Capitalism Breed Inequality*?. Allen & Unwin.

[44] Smulders, S. and van de Klundert T., 1995. Imperfect competition, concentration and growth with firm-specific R&D. *European Economic Review*, 39, 139-160.

[45] Zweimuller, J., 2000. Schumpeterian entrepreneurs meet Engel's law: The impact of inequality on innovation-driven growth. *Journal of Economic Growth*, 5, 185-206.

Appendix A

Proof of Lemma 1. The current-value Hamiltonian for monopolistic firm i is given by (10). To introduce the upper bound μ on price $p_t(i)$, we modify (10) as follows:[37]

$$H_t(i) = \Pi_t(i) - I_t(i) + \eta_t(i)\dot{Z}_t(i) + \xi_t(i)[\mu - p_t(i)], \tag{10'}$$

where $\xi_t(i)$ is the multiplier on $p_t(i) \leq \mu$. Substituting (6)-(8) into (10'), we can derive

$$\frac{\partial H_t(i)}{\partial p_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial p_t(i)} = \xi_t(i), \tag{A1}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \eta_t(i) = 1, \tag{A2}$$

$$\frac{\partial H_t(i)}{\partial Z_t(i)} = \alpha \left\{ [p_t(i) - 1] \left[\frac{\theta}{p_t(i)}\right]^{1/(1-\theta)} L_t^{\gamma}(i) R^{1-\gamma} - \phi \right\} \frac{Z_t^{1-\alpha}}{Z_t^{1-\alpha}(i)} = r_t \eta_t(i) - \dot{\eta}_t(i). \tag{A3}$$

If $p_t(i) < \mu$, then $\xi_t(i) = 0$; in this case, $\partial \Pi_t(i)/\partial p_t(i) = 0$ yields $p_t(i) = 1/\theta$. If the constraint on $p_t(i)$ is binding, then $\xi_t(i) > 0$; in this case, $p_t(i) = \mu$. Given $\mu < 1/\theta$, we have $p_t(i) = \mu$. We use (A2), (13) and $p_t(i) = \mu$ in (A3) and impose symmetry for (14). ∎

Proof of Lemma 2. Substituting (16) into $a_t = N_t V_t + v_t R$ yields

$$a_t = N_t \beta X_t + v_t R = (\theta/\mu)\beta Y_t + v_t R, \tag{A4}$$

where the second equality uses $\theta Y_t = N_t(\mu X_t)$.[38] Differentiating (A4) with respect to t yields

$$(\theta/\mu)\beta \dot{Y}_t + \dot{v}_t R = \dot{a}_t = r_t[(\theta/\mu)\beta Y_t + v_t R] + w_t L_t - c_t, \tag{A5}$$

where the second equality uses (29) and (A4). Using (3) for r_t, (4) for ω_t, $r_t v_t = \omega_t + \dot{v}_t$ and $\gamma(1-\theta)Y_t = w_t L_t$, we can rearrange (A5) to obtain

$$\frac{\dot{c}_t}{c_t} - \frac{\dot{Y}_t}{Y_t} = \frac{\mu}{\theta\beta}\frac{c_t}{Y_t} - \left[\rho + \frac{\mu(1-\theta)}{\theta\beta}\right], \tag{A6}$$

where the right-hand side is increasing in c_t/Y_t with a strictly negative y-intercept. Therefore, c_t/Y_t must jump to the steady-state value in (21). ∎

Proof of Proposition 1. In the pre-industrial era, the variety growth rate n_t is zero; in this case, the dynamics of x_t in (25) is given by $\dot{x}_t = \gamma\lambda x_t$. Equation (26a) shows that when $x_t > x_N$, variety innovation occurs (i.e., $n_t > 0$); in this case, we substitute (26a) into (25) to derive the dynamics of x_t in the first industrial era. In the second industrial era (i.e., $x_t > x_Z$), quality innovation also occurs (i.e., $z_t > 0$); in this case, we can substitute (26b) into (24) and set $z_t = 0$ to derive the following threshold:

$$x_Z \equiv \underset{x}{\arg\text{solve}} \left\{ \left[\frac{\mu - 1}{\mu^{1/(1-\theta)}}x - \phi\right] \left[\alpha - \frac{\mu^{1/(1-\theta)}(1-\gamma)}{\beta x}\right] = \gamma(\rho + \lambda) \right\}.$$

[37] Note that $L_t(i)$ is not chosen by the monopolistic firm i, which takes its equilibrium level in (6) as given.

[38] We derive this by using $p_t(i) = \mu$ and $X_t(i) = X_t$ for $\theta Y_t = \int_0^{N_t} p_t(i) X_t(i) di$.

We can also substitute (24) into (26b) to derive

$$n_t = \frac{\left[(1-\alpha)(\mu-1)-\rho\beta\right] x_t/\mu^{1/(1-\theta)} - (1-\alpha)\phi + \rho + \gamma\lambda}{\beta x_t/\mu^{1/(1-\theta)} - (1-\gamma)}. \tag{A7}$$

Substituting (A7) into (25) yields

$$\dot{x}_t = \frac{\gamma}{\beta - (1-\gamma)\mu^{1/(1-\theta)}/x_t} \left(d_1 - d_2 x_t\right), \tag{A8}$$

where we define

$$d_1 \equiv \mu^{1/(1-\theta)} \left[(1-\alpha)\phi - \lambda - \rho\right], \tag{A9a}$$

$$d_2 \equiv \beta \left[\frac{(1-\alpha)(\mu-1)}{\beta} - \lambda - \rho\right]. \tag{A9b}$$

We approximate $(1-\gamma)\mu^{1/(1-\theta)}/x_t \approx 0$ in (A8). The resulting linearized dynamics of x_t has a unique steady state that is stable if $d_1 > 0$ and $d_2 > 0$ from which we obtain $\rho + \lambda < \min\{(1-\alpha)\phi, (1-\alpha)(\mu-1)/\beta\}$. Then, $\dot{x}_t = 0$ yields $x^* = d_1/d_2$ in (27), and we impose parameter restrictions to ensure $x^* > x_Z$. Finally, substituting (27) into (23) yields (28). ■

Proof of Lemma 3. Income received by household h is given by

$$y_t(h) = r_t a_t(h) + w_t L_t = r_t a_t s_{a,t}(h) + w_t L_t = r_t a_t s_R(h) + w_t L_t, \tag{A10}$$

where the identity index $h \in [0, 1]$ is uniformly distributed and ordered in an ascending order of income. The Gini coefficient of income is defined as

$$\sigma_{y,t} = 1 - 2\int_0^1 \mathcal{L}_{y,t}(h)dh, \tag{A11}$$

where $\mathcal{L}_{y,t}(h)$ is the Lorenz curve of income. $\mathcal{L}_{y,t}(h)$ is given by

$$\mathcal{L}_{y,t}(h) \equiv \frac{\int_0^h y_t(\chi)d\chi}{\int_0^1 y_t(\chi)d\chi} = \frac{r_t a_t \int_0^h s_R(\chi)d\chi + w_t L_t \int_0^h 1 d\chi}{r_t a_t + w_t L_t}, \tag{A12}$$

where $\int_0^h 1d\chi = h$ and $\int_0^h s_R(\chi)d\chi$ is the Lorenz curve $\mathcal{L}_a(h)$ of wealth. Substituting (A12) into (A11) yields

$$\sigma_{y,t} = 1 - \frac{2r_t a_t}{r_t a_t + w_t L_t}\left[\int_0^1 \mathcal{L}_a(h)dh + \frac{w_t L_t}{r_t a_t}\int_0^1 hdh\right], \tag{A13}$$

where $\int_0^1 hdh = 0.5$. Substituting the Gini coefficient of wealth $\sigma_a \equiv 1 - 2\int_0^1 \mathcal{L}_a(h)dh$ into (A13) yields the Gini coefficient of income in (39). ■

Proof of Proposition 3. Using (3) and $\gamma(1-\theta)Y_t = w_t L_t$, we obtain

$$\frac{r_t a_t}{w_t L_t} = \frac{\rho + g_t}{\gamma(1-\theta)}\frac{a_t}{Y_t} = \frac{\rho + g_t}{\gamma(1-\theta)}\frac{a_t}{c_t}\frac{c_t}{Y_t}. \tag{A14}$$

Substituting (21) and (31) into (A14) yields

$$\frac{r_t a_t}{w_t L_t} = \frac{\rho + g_t}{\gamma\rho}\left(\frac{\rho}{\mu}\frac{\beta\theta}{1-\theta} + 1 - \gamma\right). \tag{A15}$$

Finally, we substitute (A15) into (39) to derive (42). ■