Forward compositions of inner functions

Note Title [redacted]

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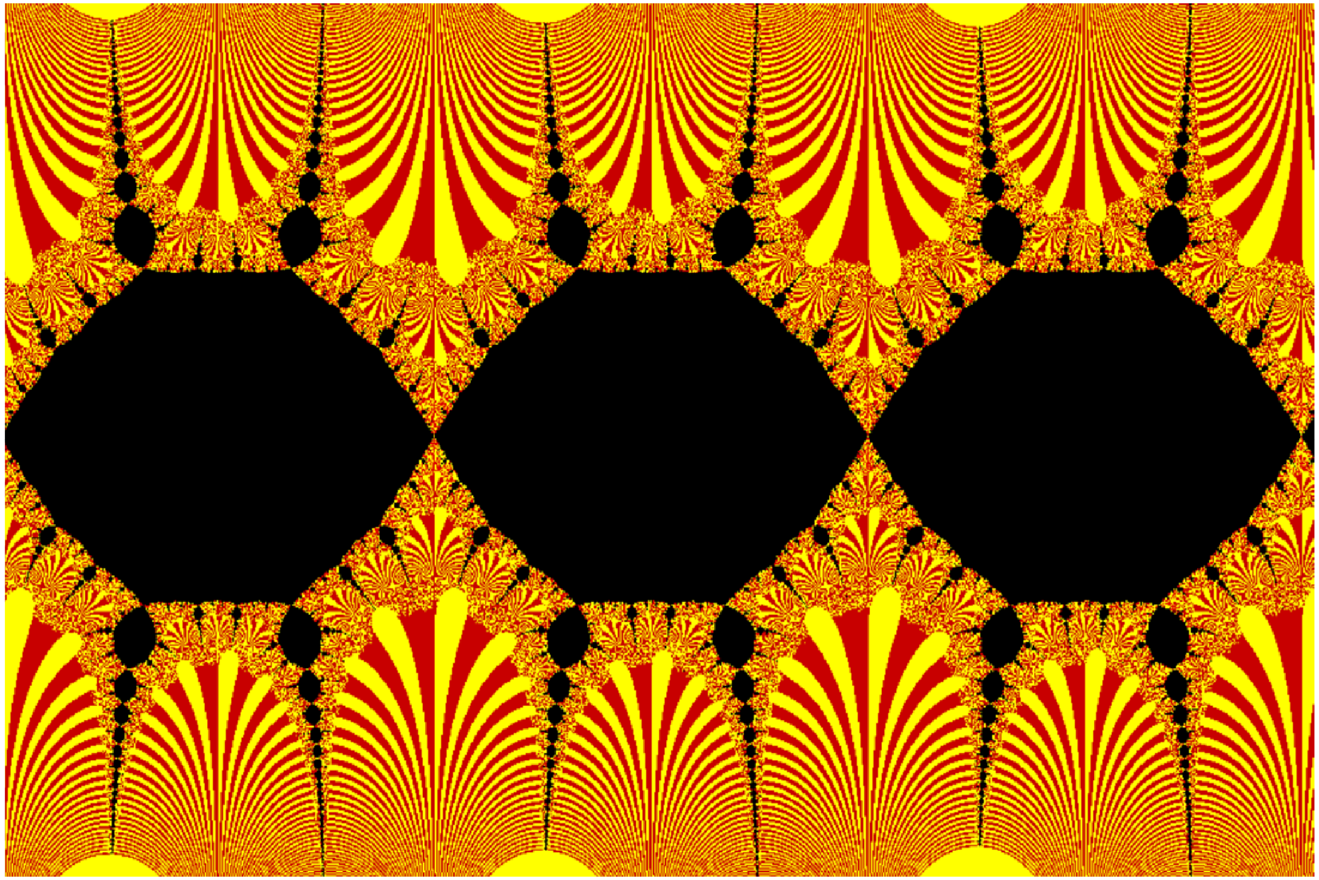

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Definition Let $f:\mathbb{D}\to\mathbb{D}$ be holomorphic. Then f is an *inner function* if the radial limits of f on $\partial\mathbb{D}$ almost all lie on $\partial\mathbb{D}$.

eg Blaschke products

These arise naturally in holomorphic dynamics.

Let U be a simply connected Fatou component of f, a tef, and $U_n \supset f^n(U)$ be Fatou components.

$z_n = f^n(z_0)$

$$
\begin{tikzcd}
U \ni z_0 \arrow[r, "f"] & U_1 \ni z_1 \arrow[r, "f"] & \cdots \arrow[r, "f"] & U_n \ni z_n \arrow[r] & \cdots \\
\mathbb{D} \ni 0 \arrow[u, "\varphi_0"] \arrow[r, "g_1"] & \mathbb{D} \ni 0 \arrow[u, "\varphi_1"] \arrow[r, "g_2"] & \cdots \arrow[r, "g_n"] & \mathbb{D} \ni 0 \arrow[u, "\varphi_n"] \arrow[r] & \cdots
\end{tikzcd}
$$

Riemann maps

$\varphi_n(0) = z_n$

inner functions

g_n different if U is a wandering domain

$\boxed{\text{Forward composition of inner functions } f_n, \quad F_n = f_n \circ \cdots \circ f_1, \quad n \geq 1.}$

In this talk, $f_n(0) = 0$ for all n. centred

By Schwarz lemma,

$$|F_n(z)| \downarrow \quad \text{for all } z \in \mathbb{D},$$

and (F_n) contracting means $\lambda_n = |f_n'(0)|$ BEFRS22

$$|F_n(z)| \downarrow 0 \text{ as } n \to \infty \iff \lambda_1 \lambda_2 \cdots \lambda_n \to 0 \text{ as } n \to \infty$$

$$\iff \sum_{n=1}^{\infty} \mu_n = \infty. \quad \mu_n = 1 - \lambda_n$$

Corresponds to a contracting SCWD:

$$\mathrm{dist}_{U_n}(f^n(z), f^n(w)) \to 0 \text{ as } n \to \infty, \quad \text{all } z, w \in U.$$

Behaviour of (F_n) on $\partial\mathbb{D}$

Hill & Velani 1995

Definition A shrinking target is a sequence (I_n) of arcs in $\partial\mathbb{D}$, not necessarily nested, s.t. $|I_n| \to 0$ as $n \to \infty$.

$|I_n|$ = length of I_n

The sequence (F_n) hits (I_n) if $F_n(\zeta) \in I_n$, infinitely often.

Thm A (BEFRS 2024a) Let $F_n = f_n \circ \cdots \circ f_1$, f_n inner fns, $f_n(0) = 0$.

(a) $|f_n'(0)| \leq \lambda < 1$, $n \geq 1$ and $\sum_{n=1}^{\infty} |I_n| = \infty$ $\Rightarrow$ $(F_n(\zeta))$ hits (I_n), for a.e. $\zeta \in \partial\mathbb{D}$.

uniform contraction

Fernández, Melián, Pestana 2007

(b) $\sum_{n=1}^{\infty} \mu_n |I_n| = \infty$ $\Rightarrow$ $(F_n(\zeta))$ hits (I_n), for a.e. $\zeta \in \partial\mathbb{D}$.

non-uniform contraction $\Rightarrow$ a.e. orbit is dense

Pommerenke's mixing results

Let $F_n = f_n \circ \cdots \circ f_1$, f_n inner fns, $f_n(0)=0$, contracting.

Let $A \subset \partial\mathbb{D}$ be an arc, $E \subset \partial\mathbb{D}$ satisfy $|E|>0$. Then

(a) $$\left| \frac{|A \cap F_n^{-1}(E)|}{|F_n^{-1}(E)|} - \frac{|A|}{2\pi} \right| = o(1) \quad \text{as } n \to \infty;$$ $|F_n^{-1}(E)| = |E|$

(b) if, in addition, $|f_n'(0)| \le \lambda$, where $\frac{1}{2} \le \lambda < 1$, then
$$\left| \frac{|A \cap F_n^{-1}(E)|}{|F_n^{-1}(E)|} - \frac{|A|}{2\pi} \right| \le K c^n, \quad \text{where } c = \exp\left(-\frac{(1-\lambda)}{84}\right).$$

Proof of (b) uses:
$$|F_n(z)| \le \frac{\lambda^n}{(1-|z|)^{13}}, \quad \text{for } z \in \mathbb{D}. \quad (*)$$

plus Poisson integrals

Question

Can we obtain $(*)$ with λ^n replaced by $\lambda_1 \lambda_2 \cdots \lambda_n$?

Thm B Let $F_n = f_n \circ \cdots \circ f_1$, f_n inner fns, $f_n(0)=0$, $|f_n'(0)| \le \lambda_n$, where $0 < a \le \lambda_n < 1$ and $\lambda_n \cdots \lambda_1 \to 0$ as $n \to \infty$. Then

(a) $$|F_n(z)| \le \lambda_n \cdots \lambda_1 |z| \exp\left(\frac{4e/a}{1-|z|}\right), \quad \text{for } z \in \mathbb{D},$$

(b) $$|F_n(z)| \le C \lambda_n \cdots \lambda_1 \frac{|z|}{(1-|z|)^p}, \quad \text{for } z \in \mathbb{D},$$

where $p = \left[\frac{2}{a}\right] + 1$, and $C = C(p) > 0$.

$a = \frac{1}{2}$, $p = 5$

Taking $a = \frac{1}{2}$ and $p = 5$ gives the mixing result:

$$\left| \frac{|A \cap F_n^{-1}(E)|}{|F_n^{-1}(E)|} - \frac{|A|}{2\pi} \right| \le C(\lambda_1 \cdots \lambda_n)^{1/18},$$

where $A \subset \partial\mathbb{D}$ is an arc and $E \subset \partial\mathbb{D}$, $|E| > 0$, C absolute.

<u>Proof of Part (a)</u> Recall $f_n(0)=0$, $|f_n'(0)|\le\lambda_n$, $a\le\lambda_n<1$.

$$|f_n(z)| \le |z|\frac{|z|+\lambda_n}{1+\lambda_n|z|} := \psi_n(|z|)$$

So

$$|F_n(z)| \le (\psi_n\circ\cdots\circ\psi_1)(|z|)$$

$$= \Psi_n(|z|) \le |z|\prod_{k=1}^{n}\left(1-c(|z|)\mu_k\right).$$

$c(|z|)=\frac{1-|z|}{2}$

$\mu_k = 1-\lambda_k$

BEFRS22

Consider

$$G_n(z)=\frac{\Psi_n(z)}{\lambda_n\cdots\lambda_1}, \text{ so } G_{n+1}(z)=\frac{\psi_{n+1}(\Psi_n(z))}{\lambda_{n+1}\cdots\lambda_1} \text{ gives}$$

$$1<\frac{G_{n+1}(|z|)}{G_n(|z|)} = \frac{1+\Psi_{n+1}(|z|)/\lambda_{n+1}}{1+\lambda_{n+1}\Psi_{n+1}(|z|)} \le 1+\left(\frac{1-\lambda_{n+1}^2}{\lambda_{n+1}}\right)\Psi_n(|z|).$$

So

$$|G_n(|z|)| \le |z|\prod_{k=1}^{n-1}\left(1+\left(\frac{1-\lambda_{k+1}^2}{\lambda_{k+1}}\right)\Psi_k(|z|)\right)$$

$1+x\le e^{x}$

$1-x\le e^{-x}$

$$\le |z|\exp\left(\sum_{k=1}^{n}\frac{(2/a)\mu_{k+1}}{\exp\left(c(z)(\mu_1+\cdots+\mu_k)\right)}\right) \le |z|\exp\left(\frac{4e/a}{1-|z|}\right).$$

$\blacksquare$

Proof of Part (b) (idea!)

For $n > m \geq 0$, put $\Psi_{n,m} = \psi_n \circ \cdots \circ \psi_{m+1}$. Take $b \in (0,1)$.

Part (a) implies $\exists\, C = C(a,b)$ s.t. $b = b(a)$ later

$$\Psi_{n,m}(|z|) \leq C \lambda_n \cdots \lambda_{m+1} \frac{|z|}{(1-|z|)^p}, \quad \text{for } |z| \leq b. \quad (\dagger)$$

Take $r \in (b, 1)$, define $r_1 = r$, $r_{k+1} = \psi_k(r_k)$, $k = 1, 2, \ldots$.

Then $r_k \downarrow 0$ so can choose m s.t. $r_{m+1} \leq b < r_m$. By $(\dagger)$,

$$\Psi_{n,m}(r_{m+1}) \leq C \lambda_n \cdots \lambda_{m+1} \frac{r_{m+1}}{(1-r_{m+1})^p}, \quad \text{for } n > m$$

that is,

$$\Psi_{n,m-1}(r_m) \leq C \lambda_n \cdots \lambda_{m+1} \frac{\psi_m(r_m)}{(1-\psi_m(r_m))^p}.$$

Want

$$\Psi_{n,m-1}(r_m) \leq C \lambda_n \cdots \lambda_m \frac{r_m}{(1-r_m)^p},$$ then repeat to $r_{m-1}, \ldots, r_1 = r$. ∎

Use Taylor expansion of ψ_m about 1.

Thanks for your attention!