Towards Transcendental Thurston Theory

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Work in progress

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Motivation

$$f(z) = z^2 + c, c \approx -0.12 + 0.74i$$

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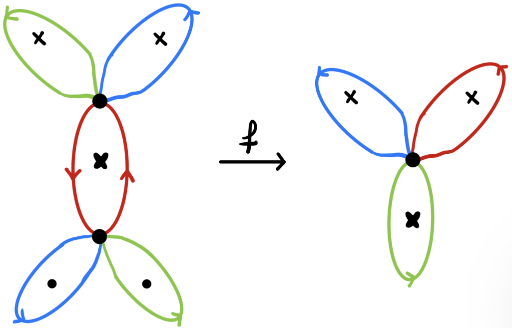

Transcendental example

$$g(z) = -\pi \cos(z)/2.$$

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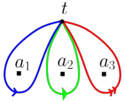
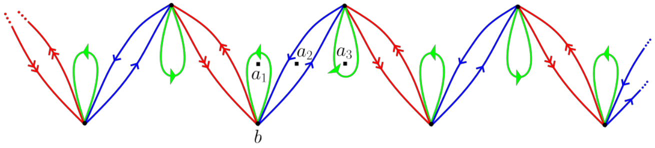
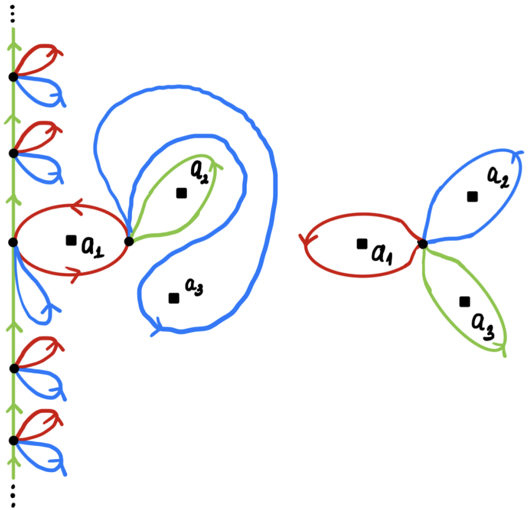




Another transcendental example



Question

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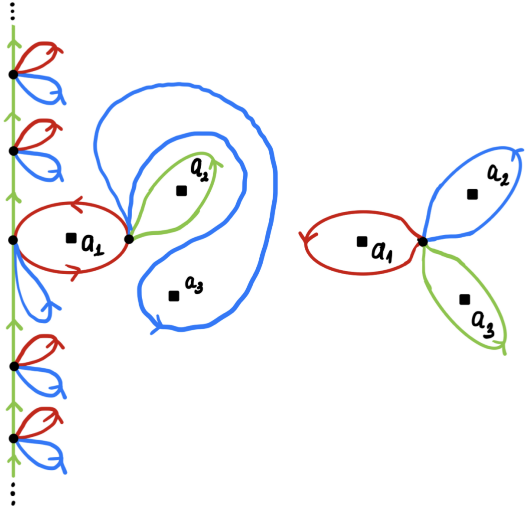
Example.

- meromorphic psf maps;
- $\varphi \circ f \circ \psi$, where f is a meromorphic psf map, $\varphi, \psi \in \mathrm{Homeo}^+(S^2)$, and $\varphi(P_f) = \psi(P_f) = P_f$.

Combinatorial model ⇝ Thurston map



Isotopies

Definition.

Let f and g be two Thurston maps with the same postsingular set P. We say that f is isotopic to g if

- $f = g \circ \varphi$ and $\varphi \in \mathrm{Homeo}^+(S^2)$;
- φ is homotopic rel. P to id_{S^2}.

Combinatorial equivalence

Definition.

Two Thurston maps f and g are combinatorially equivalence (or Thurston equivalent) if there exist two other Thurston maps $\widetilde{f}$ and $\widetilde{g}$ such that

- f and $\widetilde{f}$ are isotopic,
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- $\widetilde{f}$ and $\widetilde{g}$ are (topologically) conjugate.

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When given Thurston map is realized?

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Let f be a Thurston map with a postsingular set P_f.

Definition.

Simple closed essential curve $\gamma \subset S^2 \backslash P_f$ is called a Levy cycle for f if for some $n \geqslant 1$ there exists a simple closed curve $\gamma' \subset f^{-n}(\gamma)$ such that

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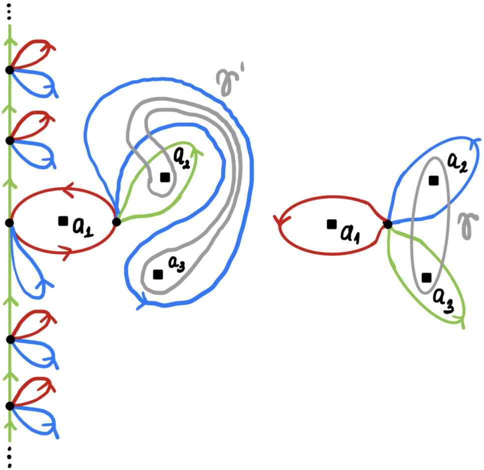
Proposition.

Suppose that a Thurston map f has a Levy cycle. Then f is obstructed.

Example of a Levy cycle



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Partial progress for structurally finite maps entire Thurston maps (S.Shemyakov'22).

In particular, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ has either finite topological degree or the only finite singular value, then it is realized if and only if f has no Levy cycles.

Main Theorem A

Theorem (NP'24).

Let f be a Thurston map such that $|P_f| = 4$. Suppose that there exists a set $A \subset P_f$ so that $|A| = 3$, $S_f \subset A$, and $|\overline{f^{-1}(A)} \cap P_f| = 3$. Then the map f is realized if and only if it has no Levy fixed curves.

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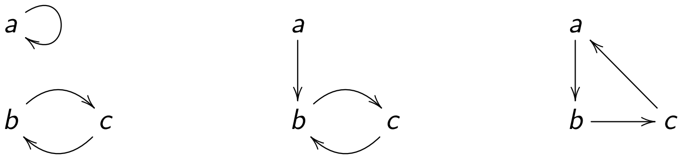

Sketch of the proof

To every Thurston map f we can associate pullback map $\sigma_f \colon \mathrm{Teich}(S^2, P_f) \circlearrowleft$, where $\mathrm{Teich}(S^2, P_f)$ is Teichmüller space of the topological sphere S^2 with the marked set P_f.

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Thurston map f is realized if and only if the pullback map σ_f has a fixed point.

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Proposition.

$\mathrm{Teich}(S^2, P_f)$ is a $(|P_f| - 3)$-complex dimensional manifold and σ_f is a holomorphic map.

Four postsingular points

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Theorem (Denjoy-Wolff theorem).

Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic map that is neither an elliptic transformation nor identity. The sequence $(g^{\circ n})$ converges uniformly on compacts to a Denjoy-Wolff point $z \in \overline{\mathbb{D}}$.

Marked Thurston maps

Let f be a Thurston map and A be a finite set such that $P_f \subset A$, and for each $a \in A$ either $f(a) \in A$ or a is an essential singularity of f. Then the pair (f, A) is called marked Thurston map and it is denote by $f\colon (S^2, A) \circlearrowleft$.

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Example.

Suppose that $f\colon S^2 \to S^2$ is a realized Thurston map having $\deg(f) + 2$ fixed points. Then the marked Thurston map $f\colon (S^2, P_f \cup \mathrm{Fix}(f)) \circlearrowleft$ is obstructed.

Main Theorem B

Theorem (NP'24).

Let $f\colon (S^2, A) \circlearrowleft$ be a marked Thurston map such that the Thurston map f is realized. Then the map $f\colon (S^2, A) \circlearrowleft$ is realized if and only if it has no Levy cycles.

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Idea of proof: If $A = P_f \cup \{a\}$ and $f(a) = a$, then the map $\sigma_{f,A}\colon \mathrm{Teich}(S^2, A) \circlearrowleft$ has a forward invariant subset U that is a properly and holomorphically embedded unit disk.

Thank you for your attention!